A SPECIAL FEATURE OF QUADRATIC MONOMIAL IDEALS

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Motivations

Throughout the talk complex = simplicial complex.

Conjecture A (Eckhoff, Kalai): The f-vector of a flag complex is the f-vector of a balanced complex.

The above conjecture has been verified by Frohmader.

Conjecture B (Kalai): The f-vector of a Cohen-Macaulay (CM) flag complex is the f-vector of a CM balanced complex.

Conjecture B is still open. The main consequence of the special feature of quadratic monomial ideals mentioned in the title is:

Theorem (Caviglia, Constantinescu, -): The h-vector of a CM flag complex is the h-vector of a CM balanced complex.

Terminology

Let Δ be a $(d-1)$-dimensional simplicial complex.

- Δ CM means $\widetilde{H}_i(\operatorname{lk} F; K) = 0$ for all $F \in \Delta$ and $i < \dim \operatorname{lk} F$.
- Δ flag means every minimal non face of Δ has cardinality 2.
- Δ balanced means that the 1-skeleton of Δ is d-colorable.

The f-vector of Δ is the vector $f(\Delta) = (f_{-1}, \ldots, f_{d-1})$ where

$$f_i = |\{i\text{-faces of } \Delta\}|$$

The h-vector of Δ is the vector $h(\Delta) = (h_0, \ldots, h_s)$ where

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}$$

Basics on the f- and h-vectors

First of all, we have the formula:

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i.$$

Therefore, if we know d, f- and h-vector determine each other. While $\dim \Delta$ can be read from the f-vector, it cannot be read from the h-vector, since s might be smaller than d.

EXAMPLE: Let Δ be the triangle $\langle \{1,2\}, \{1,3\}, \{2,3\} \rangle$. Then

$$f(\Delta) = (1,3,3) \quad \text{and} \quad h(\Delta) = (1,1,1).$$

Let Γ be the cone over Δ $\langle \{1,2,4\}, \{1,3,4\}, \{2,3,4\} \rangle$. Then

$$f(\Gamma) = (1,4,6,3) \quad \text{and} \quad h(\Gamma) = (1,1,1).$$

Basics on the f- and h-vectors

Without having information on $\dim \Delta$, if you like you can say that f- and h-vector determine each other "up to cones".

The entries of the h-vector may be negative. This is not the case if Δ is CM. A precise numerical characterization of the vectors which are h-vectors of CM complexes is due to Macaulay.

A numerical characterization of the vectors that can be obtained as f-vector of some complex has been provided by Kruskal-Katona.

As one can easily see:

$$\{f\text{-vectors of complexes}\} \subsetneq \{h\text{-vectors of CM complexes}\}$$

(for example, $(1, 3, 2, 1)$ cannot be the f-vector of any simplicial complex but is the h-vector of a CM complex).

The algebraic point of view

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, $I \subseteq S$ a homogeneous ideal and $A = S/I$ the quotient ring.

The Hilbert function of A is the map $\mathrm{HF}_A : \mathbb{N} \to \mathbb{N}$ defined as:

$$\mathrm{HF}_A(m) = \dim_K(A_m)$$

The Hilbert series of A is $\mathrm{HS}_A = \sum_{m \geq 0} \mathrm{HF}_A(m) t^m \in \mathbb{Z}[[t]]$.

If $\dim A = d$, it turns out that there is a polynomial $h(t) \in \mathbb{Z}[t]$, with $h(1) \neq 0$, such that:

$$\mathrm{HS}_A(t) = \frac{h(t)}{(1-t)^d}$$

The polynomial $h_A(t) = h(t)$ is referred as the *h*-polynomial of A.

The algebraic point of view

If Δ is on n vertices, we can consider its Stanley-Reisner (SR) ideal $I_\Delta \subseteq S$ and its SR ring $K[\Delta] = S/I_\Delta$. Notice that Δ is flag iff I_Δ is generated by quadratic monomials. A theorem of Reisner states that Δ is CM over K iff $K[\Delta]$ is a CM ring. Let us introduce also the ideal $J_\Delta = I_\Delta + (x_1^2, \ldots, x_n^2) \subseteq S$ and the ring $A_\Delta = S/J_\Delta$.

From the definitions, if $f(\Delta) = (f_{-1}, \ldots)$ and $h(\Delta) = (h_0, \ldots)$:

- $h_{K[\Delta]}(t) = \sum_i h_i t^i$,
- $h_{A_\Delta}(t) = \sum_i f_i t^i$.

Polarizing the ideal J_Δ, we get a square-free monomial ideal whose associated simplicial complex is an $(n-1)$-dimensional balanced CM complex, so f-vectors are h-vectors of CM balanced complex.

Actually, it is not difficult to show that:

$$\{f\text{-vectors of complexes}\} = \{h\text{-vectors of CM balanced complexes}\}$$

Looking for special S-regular sequences inside I_Δ

Therefore our purpose is equivalent to show that the h-vector of a CM flag complex Δ is the f-vector of some simplicial complex Γ.

It follows from the general ring theory that I_Δ contains $n-d$ quadratic polynomials $F_1, \ldots, F_{n-d}$ forming an S-regular sequence. If the F_i could be chosen monomials, then the red-statement could be easily deduced (in this case Γ can even be chosen flag).

EXAMPLE: $I_\Delta = (xy, xz, yz) \subseteq K[x, y, z]$ is the SR ideal of the 0-dimensional complex supported on 3 points. Obviously a regular sequence of length $n - d = 3 - 1 = 2$ consisting of degree 2 monomials cannot exist in $I_\Delta \subseteq K[x, y, z]$.

Actually, a result of Constantinescu,- implies that, provided Δ has no cone points, the green-statement is equivalent to $n \leq 2d$.

Looking for special S-regular sequences inside I_Δ

The main lemma of the present talk is the following:

LEMMA (Caviglia, Constantinescu, -): If K is infinite, any monomial ideal $I \subseteq S$ generated in degree 2 contains an S-regular sequence $F_1, \ldots, F_c$, where $c = \mathrm{ht}(I)$, such that each F_i is a product of 2 linear forms, namely $F_i = \ell_{1,i}\ell_{2,i}$.

EXAMPLE: The ideal $I = (xy, xz, yz) \subseteq K[x, y, z]$ of the previous slide contains the regular sequence $xy,\ z(x + y)$.

The analog of the lemma is false in higher degree: It is easy to check that $I = (x^2y, y^2z, xz^2) \subseteq K[x, y, z]$ does not contain any regular sequence of type $\ell_{1,1}\ell_{2,1}\ell_{3,1},\ \ \ell_{1,2}\ell_{2,2}\ell_{3,2}$.

The Eisenbud-Green-Harris conjecture

Before sketching the proof of the lemma, let us explain how it implies the h-vector version of the Kalai's conjecture (h-**Kalai**).

Conjecture (Eisenbud-Green-Harris): Let $I \subseteq S = K[x_1, \ldots, x_n]$ be a homogeneous ideal containing a regular sequence $F_1, \ldots, F_n$ of degrees $d_1 \leq \ldots \leq d_n$. Then there is a homogeneous ideal $J \subseteq S$ containing $(x_1^{d_1}, \ldots, x_n^{d_n})$ such that $\mathrm{HF}_{S/I} = \mathrm{HF}_{S/J}$.

Let Δ be a $(d-1)$-dimensional CM flag complex on n vertices. Pick a l.s.o.p. $\ell_1, \ldots, \ell_d$ for $K[\Delta]$ and go modulo by them. We have that $K[\Delta]/(\ell_1, \ldots, \ell_d) \cong K[y_1, \ldots, y_{n-d}]/I$ where I is a quadratic ideal, in particular I contains a regular sequence of quadrics $F_1, \ldots, F_{n-d}$. If the EGH conjecture was true, then $h(\Delta)$ would be the Hilbert function of $K[y_1, \ldots, y_{n-d}]/J$ where $J \supseteq (y_1^2, \ldots, y_{n-d}^2)$, therefore the f-vector of some simplicial complex.

The Eisenbud-Green-Harris conjecture

Unfortunately, the EGH conjecture is widely open. But.........

Theorem (Abedelfatah): With the notation of the above slide, if the regular sequence $F_1, \ldots, F_n$ in I consists of products of linear forms (i. e. $F_i = \ell_{1,i} \cdots \ell_{d_i,i}$), then the EGH conjecture holds for I.

Since the property of containing a regular sequence of products of linear forms is preserved going modulo regular linear forms, the above theorem and the lemma yield at once h-**Kalai**.

By exploiting ideas of Caviglia-Maclagan and Caviglia-Sbarra, we are able to prove a stronger statement than h-**Kalai**, however for this talk we'll be satisfied with the hereby version.

Sketch of the proof of the lemma

Take $\mathfrak{p} = (x_1, \ldots, x_c)$ a minimal prime of I of height c and write the degree 2 part of I as:

$$I_2 = x_1 V_1 \oplus \ldots \oplus x_c V_c$$

where $V_i = \langle x_j : x_i x_j \in I \text{ and } j \geq i \rangle$. For $\ell_i \in V_i$, it is easy to see that $x_1\ell_1, \ldots, x_c\ell_c$ form a regular sequence iff $\forall\ A \in \{1, \ldots, c\}$,

$$\dim_K(\langle \ell_i : i \in A\rangle \oplus \langle x_j : j \notin A\rangle) = c$$

To choose such ℓ_i consider, for each $A \in \{1, \ldots, c\}$, the bipartite graph G_A on vertices $\{1, \ldots, c\} \cup \{x_1, \ldots, x_n\}$ and edges $\{i, x_j\}$ where $i \in A$ and $j \in V_i$ or $i \notin A$ and $j = i$.

Sketch of the proof of the lemma

One shows that, since $\mathrm{ht}(I) = c$, the graph G_A satisfies the hypotheses of the Marriage Theorem, thus we can choose a perfect matching $\{1, j_1^A\}, \ldots, \{c, j_c^A\}$. This implies that

$$\dim_K \langle x_{j_1^A}, \ldots, x_{j_c^A} \rangle = c.$$

We conclude by setting as ℓ_i a general linear combination of $x_{j_i^A}$ where A runs over the subsets of $\{1, \ldots, c\}$ containing i. $\square$

A more precise conjecture

Two years ago, Constantinescu,- formulated the following more precise conjecture on the h-vectors of CM flag complexes:

$\{h\text{-vectors of CM flag complexes}\} = \{f\text{-vectors of flag complexes}\}$

By using Frohmader's solution of Eckhoff-Kalai's Conjecture 1 and a result of Björner, Frankl and Stanley, one will see that the above conjecture would yield Conjecture 2 of Kalai. To see that the right hand side is contained in the left one is a standard trick and the evidence for the reverse inclusion consist in SR ideals I_Δ of CM flag complexes on n vertices (without cone points) in the below list:

- $h(\Delta) = (h_0, h_1, h_2)$.
- $2 \cdot \mathrm{ht}(I_\Delta) \leq n$ (e. g. I_Δ is the edge ideal of a bipartite graph).
- Lots of other instances (Constantinescu,-).

Effects on the dual graph of a flag complex

Given a pure simplicial complex Δ, its dual graph $G(\Delta)$ is the simple graph whose vertices are the facets of Δ, and two facets are adjacent if and only if they have a codimension 1 face in common.

Hirsch Conjecture: If Δ is the boundary of a d-dimensional polytope on n-vertices, then $\mathrm{diam}(G(\Delta)) \leq n - d$.

Santos gave a counterexample to the above conjecture, however:

Theorem (Adiprasito-Benedetti): If Δ is a $(d-1)$-dimensional CM *flag* complex on n-vertices, then $\mathrm{diam}(G(\Delta)) \leq n - d$.

Effects on the dual graph of a flag complex

Let Δ be a pure $(d-1)$-dimensional flag complex on n vertices. The lemma in the talk implies that $G(\Delta)$ is an induced subgraph of the minimal primes-graph of a complete intersection of type:

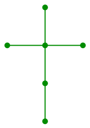
$$(x_1\ell_1, \ldots, x_{n-d}\ell_{n-d}).$$

Such graph is quite simple: It is obtained by contracting some edges (which ones depends on the geometry of the matroid given by $x_1, \ell_1, \ldots, x_{n-d}, \ell_{n-d}$) of the graph $\mathbb{G}$ such that:

- $V(\mathbb{G}) = 2^{\{1,\ldots,n-d\}}$.
- $\{A, B\} \in E(\mathbb{G})$ iff $|A \cup B| - |A \cap B| = 1$.

Effects on the dual graph of a flag complex

The fact described in the above slide puts some nontrivial restrictions for the dual graph of a pure $(d-1)$-dimensional flag complex on n vertices. For example, one can see that the graph



cannot be the dual graph of any pure $(d-1)$-dimensional flag complex on $d+3$ vertices.

By adding the CM hypothesis, we can get further rigidity by using Adiprasito-Benedetti theorem. With Benedetti we are working to find a general rigidity statement, also in the wilder setting of quadratic ideals (possibly not monomials), where we are studying the analog of the (flag) Hirsch Conjecture ....