F-thresholds of determinantal objects

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F-pure threshold

- $\Bbbk$ F-finite field of characteristic $p > 0$;
- S a standard graded polynomial ring over $\Bbbk$;
- $R = S/J$ an F-pure ring where $J \subseteq S$ is a homogeneous ideal;
- $\mathfrak{m} \subseteq S$ the unique homogeneous maximal ideal of S.

Given a homogeneous ideal $\mathfrak{a} \subseteq R$, choose a homogeneous ideal $I \subseteq S$ containing J such that $\mathfrak{a} = I/J$. For any $e \in \mathbb{N}$ set:

$$\nu_e(\mathfrak{a}) = \max\{r \in \mathbb{N} : I^r(J^{[q]} : J) \not\subseteq \mathfrak{m}^{[q]}\}, \quad q = p^e.$$

The F-**pure threshold of** $\mathfrak{a}$ is the real number

$$\operatorname{fpt}(\mathfrak{a}) = \lim_{e \to \infty} \frac{\nu_e(\mathfrak{a})}{p^e}.$$

Rem. If $J = (0)$ (so that $R = S$ and $\mathfrak{a} = I$), we have

$$\nu_e(I) = \max\{r \in \mathbb{N} : I^r \not\subseteq \mathfrak{m}^{[q]}\}, \quad q = p^e.$$

F-pure threshold

Let X be an $m \times n$ matrix of indeterminates over $\mathbb{k}$, and $S = \mathbb{k}[X]$.

Miller-Singh-_ (2014) (I)

If $J = (0)$ and I is the ideal generated by the t-minors of X, then

$$\operatorname{fpt}(I) = \min\left\{\frac{(m-k+1)(n-k+1)}{t-k+1} : k = 1, \ldots, t\right\}.$$

Singh-Takagi-_ (2016) (II)

If J is the ideal generated by the t-minors of X and $\mathfrak{a} = \mathfrak{m}/J$, then

$$\operatorname{fpt}(\mathfrak{a}) = \min\{m, n\}(t-1).$$

A crucial ingredient for proving (II) is the description given by Bruns of the canonical class in the divisor class group of a determinantal ring.

Concerning (I), let us denote by I_t the ideal generated by the t-minors of X for any $t = 1, \ldots, \min\{m, n\}$. A main tool for the proof has been:

Bruns (1991)

For all $t = 1, \ldots, \min\{m, n\}$ and $s \in \mathbb{N}$ we have

$$\overline{I_t^s} = \bigcap_{k=1}^{t} I_k^{(s(t-k+1))}.$$

From now on $J = (0)$, i.e. $R = S$ and $\mathfrak{a} = I$. If $q = p^e$, the qth root of I, denoted by $I^{[1/q]}$ is the smallest ideal $H \subseteq S$ such that

$$I \subseteq H^{[q]}.$$

If λ is a positive real number, it is readily seen that

$$\left(I^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \left(I^{\lceil \lambda p^{e+1} \rceil}\right)^{[1/p^{e+1}]} \quad \forall\ e \in \mathbb{N}.$$

The *generalized test ideal of I* with coefficient λ is

$$\tau(\lambda \cdot I) \underset{e \gg 0}{=} \left(I^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}.$$

Note that $\tau(\lambda \cdot I) \supseteq \tau(\mu \cdot I)$ whenever $\lambda \leq \mu$.

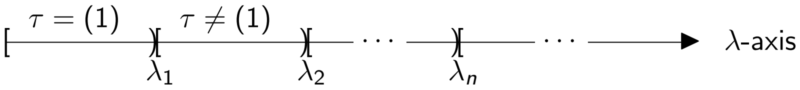
Generalized test ideals

One can also show that, $\forall\ \lambda \in \mathbb{R}_{>0}$, $\exists\ \varepsilon \in \mathbb{R}_{>0}$ such that

$$\tau(\lambda \cdot I) = \tau(\mu \cdot I) \ \ \forall\ \mu \in [\lambda, \lambda + \varepsilon).$$

A $\lambda \in \mathbb{R}_{>0}$ is called an *F-jumping number* for I if

$$\tau((\lambda - \varepsilon) \cdot I) \supsetneq \tau(\lambda \cdot I) \ \ \forall\ \varepsilon \in \mathbb{R}_{>0}.$$



$$(1) \supsetneq \tau(\lambda_1 \cdot I) \supsetneq \tau(\lambda_2 \cdot I) \supsetneq \ldots \supsetneq \tau(\lambda_n \cdot I) \supsetneq \ldots$$

The λ_i above are the F-jumping numbers. Note that $\lambda_1 = \mathrm{fpt}(I)$.

Let $X = (x_{ij})$ be an $m \times n$-generic matrix (assume $m \leq n$) and $S = \Bbbk[X]$. For $k = 1, \ldots, m$, the ideal of S generated by the k-minors of X will be denoted by I_k.

For $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{N}^m$, we denote by I^σ the ideal of S

$$I_1^{\sigma_1} I_2^{\sigma_2} \cdots I_m^{\sigma_m}.$$

More generally, if $\Sigma \subseteq \mathbb{N}^m$, we set

$$I(\Sigma) = \sum_{\sigma \in \Sigma} I^\sigma \subseteq S$$

Fix $\sigma = (\sigma_1, \dots, \sigma_m) \in \mathbb{N}^m$ and $\Sigma \subseteq \mathbb{N}^m$.

- By $\Delta \in S$ being a product of minors of shape σ, we mean that $\Delta = \delta_1 \cdots \delta_s$ where

$$\sigma_i = |\{j = 1, \dots, s : \delta_j \ \text{ is an } i\text{-minor of } X\}|.$$

- For $k = 1, \dots, m$, set $\gamma_k(\sigma) = \sum_{i=k}^{m} \sigma_i(i - k + 1)$.
- By $C_\Sigma \subseteq \mathbb{Q}^m$ we denote the convex hull of the set

$$\bigcup_{\sigma \in \Sigma} \left((\gamma_1(\sigma), \gamma_2(\sigma), \dots, \gamma_m(\sigma)) + \mathbb{Q}^m_{\geq 0}\right) \subseteq \mathbb{Q}^m.$$

Henriques-_ (2016)

For $\lambda \in \mathbb{R}_{>0}$, the ideal $\tau(\lambda \cdot I(\Sigma))$ is generated by the product of minors of shape σ such that there is $(v_1, \dots, v_m) \in C_\Sigma$ for which

$$\gamma_k(\sigma) \geq \lfloor \lambda v_k \rfloor + 1 - (m - k + 1)(n - k + 1) \quad \forall\ k = 1, \dots, m$$

Where does the previous result come from?

From now on the goal will be to explain a general method to infer the previous theorem. First of all recall the following result:

De Concini-Eisenbud-Procesi (1980)

If $t \in \{1, \dots, m\}$ and $s \in \mathbb{N}$, the symbolic power $I_t^{(s)}$ is generated by the products of minors of shape σ where $\gamma_t(\sigma) \geq s$

Therefore one can see that the previous formula is equivalent to

$$\tau(\lambda \cdot I(\Sigma)) = \sum_{(v_1,\dots,v_m)\in C_\Sigma} \left(\bigcap_{k=1}^{m} I_k^{(\lfloor \lambda v_k \rfloor + 1 - (m-k+1)(n-k+1)\,\mathrm{ht}(I_k))} \right)$$

Where does the previous formula come from?

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be homogeneous prime ideals of S. For example:

(i) $\mathfrak{p}_k = I_k$ and $S = \mathbb{k}[X]$;

(ii) $\mathfrak{p}_k = (x_k)$ and $S = \mathbb{k}[x_1, \ldots, x_m]$.

For $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{N}^m$ let $J^\sigma = \mathfrak{p}_1^{\sigma_1} \cdots \mathfrak{p}_m^{\sigma_m}$. Also, for all k in $\{1, \ldots, m\}$, let $e_k(\sigma)$ be the maximum natural number ℓ such that

$$J^\sigma \subseteq \mathfrak{p}_k^{(\ell)}.$$

Of course we have

$$J^\sigma \overline{J^\sigma} \subseteq \bigcap_{k=1}^{m} \mathfrak{p}_k^{(e_k(\sigma))}.$$

Condition A

We say that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy A if

$$\overline{J^\sigma} = \bigcap_{k=1}^{m} \mathfrak{p}_k^{(e_k(\sigma))} \quad \forall\ \sigma \in \mathbb{N}^m.$$

For example, the ideals in (ii) obviously satisfy A, where

$$e_k(\sigma) = \sigma_k.$$

Bruns proved that also the ideals in (i) satisfy A, where

$$e_k(\sigma) = \gamma_k(\sigma).$$

Condition A+

If I and J are two ideals of S, of course

$$\overline{I^2 + J^2} \supseteq I^2 + IJ + J^2,$$

(since $(IJ)^2 \subseteq (I^2 + J^2)^2$). This generalizes as follows: if $\Sigma \subseteq \mathbb{N}^m$ and $J(\Sigma) = \sum_{\sigma \in \Sigma} J^\sigma$:

$$\overline{J(\Sigma)} \supseteq J(\overline{\Sigma}),$$

where $\overline{\Sigma}$ denotes the set of integral vectors of the convex hull of

$$\bigcup_{\sigma \in \Sigma} \left(\sigma + \mathbb{Q}^m_{\geq 0}\right) \subseteq \mathbb{Q}^m.$$

Condition A+

If $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy A, then

$$\overline{J(\Sigma)} \supseteq \sum_{\sigma \in \overline{\Sigma}} \left(\bigcap_{k=1}^{m} \mathfrak{p}_k^{(e_k(\sigma))} \right).$$

One can show that this is equivalent to the fact that

$$\overline{J(\Sigma)} \supseteq \sum_{(v_1,\ldots,v_m) \in \mathbb{Z}^m \cap K_\Sigma} \left(\bigcap_{k=1}^{m} \mathfrak{p}_k^{(v_k)} \right),$$

where $K_\Sigma \in \mathbb{Q}^m$ is the convex hull of

$$\bigcup_{\sigma \in \Sigma} \left((e_1(\sigma), e_2(\sigma), \ldots, e_m(\sigma)) + \mathbb{Q}^m_{\geq 0} \right) \subseteq \mathbb{Q}^m.$$

We say that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy A+ if

$$\overline{J(\Sigma)} = \sum_{(v_1,\ldots,v_m) \in \mathbb{Z}^m \cap K_\Sigma} \left(\bigcap_{k=1}^{m} \mathfrak{p}_k^{(v_k)} \right) \quad \forall\ \Sigma \subseteq \mathbb{N}^m.$$

Condition A+ is satisfied by

- the ideals in (i): in this case, by mixing arguments of Bruns and De Concini-Eisenbud-Procesi one has that

$$\overline{I(\Sigma)} = \sum_{(v_1,\ldots,v_m)\in\mathbb{Z}^m\cap C_\Sigma} \left(\bigcap_{k=1}^{m} I_k^{(v_k)} \right) \quad \forall\ \Sigma \subseteq \mathbb{N}^m.$$

- the ideals in (ii): in this case, if P_Σ is the Newton polyhedron of the ideal $J(\Sigma) = (x_1^{\sigma_1} \cdots x_m^{\sigma_m} : (\sigma_1, \ldots, \sigma_m) \in \Sigma)$, work of Teissier implies:

$$\overline{J(\Sigma)} = (x_1^{v_1} \cdots x_m^{v_m} : (v_1, \ldots, v_m) \in \mathbb{Z}^m \cap P_\Sigma) \quad \forall\ \Sigma \subseteq \mathbb{N}^m.$$

Condition B

We say that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy B if there exists $f \in S$ such that:

- $\mathrm{in}_\prec(f)$ is squarefree for some monomial order $\prec$;
- $f \in \mathfrak{p}_k^{(\mathrm{ht}(\mathfrak{p}_k))}$ for all $k = 1, \ldots, m$.

The ideals in (ii) obviously satisfy B: indeed

$$f = x_1 \cdots x_m \in (x_k) = \mathfrak{p}_k^{(\mathrm{ht}(\mathfrak{p}_k))} \;\; \forall \; k = 1, \ldots, m.$$

Condition B

Also the ideals in (i) satisfy B:

$$X = \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{pmatrix}$$

Let f be the product of the diagonal minors, and $\prec$ the lex with

$$x_{11} > x_{12} > \ldots > x_{1n} > x_{21} > x_{22} > \ldots > x_{2n} > \ldots > x_{mn}.$$

Then $\mathrm{in}_{\prec}(f) = \prod_{(i,j)} x_{ij}$. By looking at the shape of f, using the mentioned result of De Concini-Eisenbud-Procesi one checks that

$$f \in I_k^{((m-k+1)(n-k+1))} = \mathfrak{p}_k^{(\mathrm{ht}(\mathfrak{p}_k))} \quad \forall\ k = 1, \ldots, m.$$

The general result

Henriques-_ (2016)

If $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy A and B, then $\forall\ \sigma \in \mathbb{N}^m$,

$$\tau(\lambda \cdot J^\sigma) = \bigcap_{k=1}^{m} \mathfrak{p}_k^{(\lfloor \lambda e_k(\sigma) \rfloor + 1 - \mathrm{ht}(\mathfrak{p}_k))} \quad \forall\ \lambda \in \mathbb{R}_{>0}.$$

If $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy A+ and B, then $\forall\ \Sigma \subseteq \mathbb{N}^m$,

$$\tau(\lambda \cdot J(\Sigma)) = \sum_{(v_1,\ldots,v_m) \in K_\Sigma} \left(\bigcap_{k=1}^{m} \mathfrak{p}_k^{(\lfloor \lambda v_k \rfloor + 1 - \mathrm{ht}(\mathfrak{p}_k))} \right) \quad \forall\ \lambda \in \mathbb{R}_{>0}.$$

Problem

Find a natural class of finite sets of prime ideals satisfying $A+$ and B containing both determinantal ideals and principal ideals generated by variables.

THANKS FOR YOUR ATTENTION