AN INTRODUCTION TO $\mathcal{D}$-MODULES

WITH APPLICATIONS TO LOCAL COHOMOLOGY

Motivations

Craig Huneke, in a *Research Notes in Mathematics* of 1992, raised the following question:

R commutative noetherian ring, $I \subseteq R$ an ideal. If M is a finitely generated (f.g.) R-module, is $\mathrm{Ass}(H^i_I(M))$ a finite set for all i?

Motivations

The point of the question above is that $H^i_I(M)$ may be not finitely generated. In some cases however the answer is clearly positive:

(i) M f.g. $\Rightarrow H^0_I(M)$ f.g.; in particular, $\mathrm{Ass}(H^0_I(M))$ is finite.

(ii) If $I = \mathfrak{m}$ is maximal, then $\mathrm{Ass}(H^i_{\mathfrak{m}}(M)) \subseteq \{\mathfrak{m}\}$ for all i.

Motivations

A bit of history of the problem...

(i) Huneke-Sharp, *Trans. Amer. Math. Soc. 1993*: If R is regular and $\mathrm{char}(R) > 0$, then then for any ideal $|\operatorname{Ass}(H_I^i(R))| < \infty$.

(ii) Lyubeznik, *Invent. Math. 1993*: If R is a regular *local* ring containing $\mathbb{Q}$, then for any ideal $|\operatorname{Ass}(H_I^i(R))| < \infty$.

(iii) Lyubeznik, *Comm. Algebra 2000*: If R is an unramified regular *local* ring of mixed characteristic, then for any ideal $|\operatorname{Ass}(H_I^i(R))| < \infty$.

(iv) Bhatt-Blickle-Lyubeznik-Singh-Zhang, *2012*: If $R = \mathbb{Z}[x_1, \ldots, x_n]$, then for any ideal $|\operatorname{Ass}(H_I^i(R))| < \infty$.

Motivations

(iv) Brodmann-Lashgari, *Proc. Amer. Math. Soc. 2000*: If g is grade(I, M), then $\operatorname{Ass}(H_I^1(M))$ and $\operatorname{Ass}(H_I^g(M))$ are finite.

(v) Singh, *Math. Res. Lett. 2000*: If $R = \dfrac{\mathbb{Z}[u, v, w, x, y, z]}{(ux + vy + wz)}$ and $I = (x, y, z) \subseteq R$, then $H_I^3(R)$ has p-torsion elements for infinitely many primes. Consequently $\operatorname{Ass}(H_I^3(R))$ is infinite.

(vi) Katzman, *J. Algebra 2002*: For any field K, if R is the ring $\dfrac{K[s, t, u, v, x, y]}{(su^2x^2 + (s + t)uvxy + tv^2y^2)}$, then $|\operatorname{Ass}(H^2_{(x,y)}(R))| = \infty$.

(vii) Takagi-Takahashi, *Math. Res. Lett. 2007*: If R has positive characteristic, is Gorenstein and has F-finite representation type, then for any ideal $|\operatorname{Ass}(H_I^i(R))| < \infty$.

Motivations

In the proof of Huneke-Sharp (R regular of positive characteristic), it is crucial that the iterated Frobenius maps $F^e : R \to R$, sending r to r^{p^e}, are flat and that $\{F^e(I)R\}_{e\in\mathbb{N}}$ is cofinal with $\{I^k\}_{k\in\mathbb{N}}$.

A family of ring homomorphisms with these two properties does not exist for $R = \mathbb{Q}[[x_1, \ldots, x_n]]$. In these lectures we will explore the machinery used by Lyubeznik to prove his first theorem:

If R is a regular local ring containing $\mathbb{Q}$ and $I \subseteq R$ is an ideal, then $\mathrm{Ass}(H_I^i(R))$ is finite for every i.

To this aim, we need to introduce the theory of $\mathcal{D}$-modules.

In these seminars I'll follow some lectures given by Anurag Singh at the Local Cohomology Workshop held in Mumbai (India) in 2011.

Some almost commutative algebra

Let A be a (possibly noncommutative) ring. A left A-module M is *noetherian* if one of the following equivalent conditions holds:

(i) All submodules are finitely generated.

(ii) ACC holds on submodules.

(iii) ACC holds on f.g. submodules.

EXAMPLE: The free K-algebra $A = K\langle x, y\rangle$ on x and y is not noetherian: In fact the left ideal $\sum_{i\in\mathbb{N}} Axy^i$ is not f.g..

Some almost commutative algebra

Let K be a field in the center of A.

A *filtration* on A is a family $\mathcal{F} = \{\mathcal{F}_i\}_{i \geq -1}$ such that:

(i) $\mathcal{F}_i$ are finite-dimensional K-vector spaces;

(ii) $\mathcal{F}_{-1} = \{0\}$ and $1 \in \mathcal{F}_0$;

(iii) $\mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq A$ and $\bigcup_{i \geq -1} \mathcal{F}_i = A$.

(iv) $\mathcal{F}_i \cdot \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$.

From now on we will fix the filtration $\mathcal{F}$ and assume that $\mathrm{gr}^{\mathcal{F}} A = \bigoplus_{i \geq 0} \mathcal{F}_i / \mathcal{F}_{i-1}$ is a commutative noetherian ring.

Such rings are called *almost commutative*.

Some almost commutative algebra

Let M be a left A-module.

A *filtration* on M is a family $\mathcal{G} = \{\mathcal{G}_i\}_{i \geq m}$, $m \in \mathbb{Z}$, such that:

(i) $\mathcal{G}_i$ are finite $\mathcal{F}_0$-modules;

(ii) $\mathcal{G}_m = \{0\}$;

(iii) $\mathcal{G}_m \subseteq \mathcal{G}_{m+1} \subseteq \mathcal{G}_{m+2} \subseteq \ldots \subseteq M$ and $\bigcup_{i \geq m} \mathcal{G}_i = M$.

(iv) $\mathcal{F}_i \cdot \mathcal{G}_j \subseteq \mathcal{G}_{i+j}$.

We will denote by $\mathrm{gr}^{\mathcal{G}} M$ the $\mathrm{gr}^{\mathcal{F}} A$-module $\bigoplus_{i \geq m+1} \mathcal{G}_i / \mathcal{G}_{i-1}$. $\mathcal{G}$ is called a *good filtration* if $\mathrm{gr}^{\mathcal{G}} M$ is finitely generated over $\mathrm{gr}^{\mathcal{F}} A$.

REMARK: If M is finitely generated as a left A-module by $m_1, \ldots, m_k$, then $\mathcal{G}_s = \sum_{i=1}^{k} \mathcal{F}_s m_i$ is a good filtration on M.

Some almost commutative algebra

LEMMA: Let M be a left A-module, and $\mathcal{G}$ and $\mathcal{H}$ filtrations on it.

(i) $\mathcal{G}$ is good if and only if there is a $j_0 \in \mathbb{Z}$ such that:

$$\mathcal{F}_i \cdot \mathcal{G}_j = \mathcal{G}_{i+j} \ \forall \ i \geq 0, \ j \geq j_0.$$

(ii) If $\mathcal{G}$ is good, then M is finitely generated over A.

(iii) If $\mathcal{G}$ is good, then there exists $r \in \mathbb{Z}$ such that:

$$\mathcal{G}_t \subseteq \mathcal{H}_{t+r} \ \forall \ t \in \mathbb{Z}.$$

If also $\mathcal{H}$ is good, then there exists $s \in \mathbb{Z}$ such that:

$$\mathcal{H}_{t-s} \subseteq \mathcal{G}_t \subseteq \mathcal{H}_{t+s} \ \forall \ t \in \mathbb{Z}.$$

Some almost commutative algebra

Proof: (i). Put $\widetilde{\mathcal{F}_j} = \bigoplus_{i \le j} \mathcal{F}_i/\mathcal{F}_{i-1}$ and $\widetilde{\mathcal{G}}_j = \bigoplus_{i \le j} \mathcal{G}_i/\mathcal{G}_{i-1}$. If $\mathcal{G}$ is good, then there exists $j_0 \in \mathbb{Z}$ such that

$$\widetilde{\mathcal{F}_i} \cdot \widetilde{\mathcal{G}}_j = \widetilde{\mathcal{G}_{i+j}} \; \forall \; i \ge 0, \; j \ge j_0$$

This means that, for every $i \ge 0, \; j \ge j_0$, $\mathcal{F}_i \cdot \mathcal{G}_j + \mathcal{G}_{i+j-1} = \mathcal{G}_{i+j}$. By inducting on i we can assume $\mathcal{G}_{i+j-1} = \mathcal{F}_{i-1} \cdot \mathcal{G}_j \;\; (1 \in \mathcal{F}_0)$. This yields $\mathcal{F}_i \cdot \mathcal{G}_j = \mathcal{G}_{i+j}$ since $\mathcal{F}_{i-1} \subseteq \mathcal{F}_i$. The converse is trivial.

(ii) follows immediately from (i).

(iii) Let $j_0 \in \mathbb{Z}$ be as in (i). Since $\mathcal{G}$ and $\mathcal{H}$ are filtrations, there exists $r \in \mathbb{Z}$ such that $\mathcal{G}_{j_0} \subseteq \mathcal{H}_r$. So

$$\mathcal{G}_t \subseteq \mathcal{G}_{t+j_0} = \mathcal{F}_t \cdot \mathcal{G}_{j_0} \subseteq \mathcal{F}_t \cdot \mathcal{H}_r \subseteq \mathcal{H}_{t+r}. \qquad \square$$

Some almost commutative algebra

Assume that $\mathcal{F}_0 = K$ and $\mathrm{gr}^{\mathcal{F}} A$ is generated in degree 1 over K. If $\mathcal{G}$ is a good filtration on a left A-module M, then $\dim_K \mathcal{G}_i/\mathcal{G}_{i-1}$ is a polynomial in i for $i \gg 0$. This implies that $\dim_K \mathcal{G}_i$ is also a polynomial in i for $i \gg 0$. Let us call such a polynomial $P_{\mathcal{G}}$.

The polynomial $P_{\mathcal{G}}$ is not an invariant of M, depending on the good filtration $\mathcal{G}$. However, given two good filtrations $\mathcal{G}$ and $\mathcal{H}$ on M, the degrees and the leading coefficients of $P_{\mathcal{G}}$ and of $P_{\mathcal{H}}$ are the same! This follows immediately from point (iii) of the lemma.

DEFINITION: Let M be a finitely generated left A-module. By considering a good filtration $\mathcal{G}$ on it, we define:

- $\dim(M) =$ degree of $\mathcal{P}_{\mathcal{G}}$.
- $e(M) = \dim(M)! \cdot$(leading coefficient of $P_{\mathcal{G}}$).

Some almost commutative algebra

LEMMA: Let M be a left A-module with a good filtration $\mathcal{G}$. Given a short exact sequence of left A-modules

$$0 \to M' \to M \to M'' \to 0,$$

set $\mathcal{G}' = \mathcal{G} \cap M'$ and $\mathcal{G}'' = \mathrm{Im}(\mathcal{G})$. Then

$$0 \to \mathrm{gr}^{\mathcal{G}'} M' \to \mathrm{gr}^{\mathcal{G}} M \to \mathrm{gr}^{\mathcal{G}''} M'' \to 0$$

is an exact sequence of $\mathrm{gr}^{\mathcal{F}} A$-modules, $\mathcal{G}'$ and $\mathcal{G}''$ are good filtrations and:

- $\dim(M) = \max\{\dim(M'), \dim(M'')\}$.
- If $\dim(M') = \dim(M'')$, then $e(M) = e(M') + e(M'')$.

In particular, if M is a f.g. left A-module, then it is noetherian.

The Weyl algebra

Let K be a field of characteristic 0 and $\mathcal{D}$ be the *Weil algebra*, namely the free K-algebra $K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ mod out by

$$\begin{cases} [x_i, x_j] = x_i x_j - x_j x_i = 0 \\ [\partial_i, \partial_j] = \partial_i \partial_j - \partial_j \partial_i = 0 \\ [\partial_i, x_j] = \partial_i x_j - x_i \partial_j = \delta_{ij} \end{cases}$$

Given two vectors $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\underline{\beta} = (\beta_1, \ldots, \beta_n)$ in $\mathbb{N}^n$, we write $\underline{x}^{\underline{\alpha}} \underline{\partial}^{\underline{\beta}}$ for $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta^n}$. It is easy to see that:

$$\{\underline{x}^{\underline{\alpha}} \underline{\partial}^{\underline{\beta}} : \underline{\alpha}, \underline{\beta} \in \mathbb{N}^n\}$$

is a K-basis of $\mathcal{D}$, known as the Poincaré-Birkhoff-Witt basis.

The Weyl algebra

The Bernstein filtration $\mathcal{F}$ on $\mathcal{D}$ is defined as $\mathcal{F}_{-1} = 0$ and

$$\mathcal{F}_s = \langle \underline{x}^{\underline{\alpha}} \underline{\partial}^{\underline{\beta}} : |\underline{\alpha}| + |\underline{\beta}| \leq s \rangle \quad \forall\, s \in \mathbb{N}.$$

Since the only nonzero bracket decreases the "degrees", $\mathrm{gr}^{\mathcal{F}} \mathcal{D}$ is commutative. More precisely:

$$\mathrm{gr}^{\mathcal{F}} \mathcal{D} = K[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n].$$

In particular, $\dim(\mathcal{D}) = 2n$ and $e(\mathcal{D}) = 1$.

The Weyl algebra

An astonishing property

THEOREM (Bernstein) Let $M \neq 0$ be a finitely generated left $\mathcal{D}$-module. Then $n \leq \dim(M) \leq 2n$.

Proof: Let $\mathcal{G}$ be a good filtration on M. We want to show that

$$\mathcal{F}_s \longrightarrow \mathrm{Hom}_K(\mathcal{G}_s, \mathcal{G}_{2s})$$

mapping f to $m \mapsto fm$ is injective for all $s \in \mathbb{N}$. If $s = 0$ it is OK.

Suppose t is the least positive integer such that exists $0 \neq f \in \mathcal{F}_t$ with $f \cdot \mathcal{G}_t = 0$. For sure there exists i such that either x_i or ∂_i occurs in f. If ∂_i occurs in f, then $[x_i, f]$ is a nonzero element of $\mathcal{F}_{t-1}$. Then there is $m \in \mathcal{G}_{t-1} \subseteq \mathcal{G}_t$ such that $[x_i, f] \cdot m \neq 0$. However $[x_i, f] \cdot m = x_i fm - fx_i m \ \in \ x_i(f \cdot \mathcal{G}_t) - f \cdot \mathcal{G}_t = 0$. If x_i occurs in f... So the maps above are injective. In particular:

$$\text{pol. of degree } 2n \sim \dim_K(\mathcal{F}_s) \leq (\dim_K(\mathcal{G}_s)) \cdot (\dim_K(\mathcal{G}_{2s}))_{\sim \text{ pol. of degree } 2\dim(M)}.$$

□

Holonomic $\mathcal{D}$-modules

A finitely generated left $\mathcal{D}$-module M is called *holonomic* if either $M = 0$ or $\dim(M) = n$.

EXAMPLES: (i) $S = K[z_1, \dots, z_n]$ is a left $\mathcal{D}$-module by putting:

$$x_i \cdot f = z_i f, \qquad \partial_i \cdot f = \frac{\partial f}{\partial z_i} \qquad \forall\ f \in S.$$

Obviously $\mathcal{G}_s = \langle f \in S : \deg(f) \leq s \rangle$ defines a good filtration on S. So S is holonomic.

(ii) Let $\mathfrak{m} = (z_1, \dots, z_m) \subseteq S$. $H^n_{\mathfrak{m}}(S)$ is a left $\mathcal{D}$-module by:

$$\partial_i^k \cdot \frac{1}{z_1 \cdots z_n} = \frac{(-1)^k k!}{z_1 \cdots z_{i-1} z_i^{k+1} z_{i+1} \cdots x_n}.$$

Clearly $\mathcal{G}_s = \left\langle \frac{1}{z_1 \cdots z_n \cdot u} : u \text{ monomial of } S, \deg(u) \leq s \right\rangle$ defines a good filtration on $H^n_{\mathfrak{m}}(S)$. In particular $H^n_{\mathfrak{m}}(S)$ is holonomic.

Holonomic $\mathcal{D}$-modules

Let $S = K[z_1, \dots, z_n]$ be the polynomial ring in n variables over K. Clearly there is a K-algebra homomorphism (indeed an inclusion):

$$\begin{array}{ccc} S & \stackrel{\iota}{\to} & \mathcal{D} \\ z_i & \mapsto & x_i \end{array}$$

From this, we can (and will) view any left $\mathcal{D}$-module M as an S-module via restriction by ι. In particular, we are allowed to define the set of associated primes over S of any $\mathcal{D}$-module M:

$$\mathrm{Ass}_S(M) = \{\mathfrak{p} \in \mathrm{Spec}(S) \ : \ \mathfrak{p} = 0 :_S m \ \text{ for some } 0 \neq m \in M\}$$

The first goal of today is to show the following:

THEOREM: If M is holonomic, then $|\mathrm{Ass}_S(M)| \leq e(M) < \infty$.

Let $f \in S$ and keep on denoting f its image $\iota(f) \in \mathcal{D}$. By using the definition of the multiplication in the Weyl algebra:

$$f \cdot \partial_i = \partial_i \cdot f - \frac{\partial f}{\partial z_i} \quad \forall\ i \in \{1, \ldots, n\}.$$

Doing an induction on s, we get

$$f^s \cdot \partial_i = \partial_i \cdot f^s - sf^{s-1}\frac{\partial f}{\partial z_i}.$$

This shows the following:

REMARK: Let M be a left $\mathcal{D}$-module and I an ideal of S. By considering M as an S-module, we can form the S-module

$$H^0_I(M) = \{m \in M : I^s m = 0 \text{ for some } s \in \mathbb{N}\}.$$

Then $H^0_I(M)$ is a $\mathcal{D}$-submodule of M. $\square$

Holonomic $\mathcal{D}$-modules

Given a shot exact sequence of left $\mathcal{D}$-modules

$$0 \to M' \to M \to M'' \to 0$$

we have that:

(i) M is holonomic if and only if M' and M'' are holonomic.

(ii) If (i) holds, then $e(M) = e(M') + e(M'')$.

PROPOSITION: If M is holonomic, then $\text{length}_{\mathcal{D}}(M) \leq e(M)$.

Proof: Take a chain of $\mathcal{D}$-modules:

$$0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_\ell = M.$$

From the exact sequences $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$, we infer that M_i and M_i/M_{i-1} are nonzero holonomic $\mathcal{D}$-modules for all $i = 1, \ldots, \ell$ and that

$$0 < e(M_1) < e(M_2) < \ldots < e(M_\ell) = e(M).$$

□

Holonomic $\mathcal{D}$-modules

We are now ready to show the following for left $\mathcal{D}$-modules M:

THEOREM: If M is holonomic, then $|\operatorname{Ass}_S(M)| \leq e(M) < \infty$.

Proof: We want to induct on $\ell = \text{length}_{\mathcal{D}}(M)$ ($\leq e(M)$). If $\ell = 1$, take $\mathfrak{p} \in \operatorname{Ass}_S(M)$. Then $H^0_{\mathfrak{p}}(M)$ is a nonzero $\mathcal{D}$-submodule of M. Because $\ell = 1$, then we have $H^0_{\mathfrak{p}}(M) \cong M$. If $\mathfrak{q} \in \operatorname{Ass}_S(M) = \operatorname{Ass}_S(H^0_{\mathfrak{p}}(M))$ one has $\mathfrak{p} \subseteq \mathfrak{q}$, so by symmetry we deduce $\mathfrak{q} = \mathfrak{p}$.

If $\ell > 1$, take a (nonzero) simple $\mathcal{D}$-submodule $N \subseteq M$. Of course $\text{length}_S(M/N) < \ell$, and by the short exact sequence of $\mathcal{D}$-modules

$$0 \to N \to M \to M/N \to 0$$

we get that N and M/N are holonomic $\mathcal{D}$-modules. However that above is also an exact sequence of S-modules, therefore we have:

$$\operatorname{Ass}_S(M) \subseteq \operatorname{Ass}_S(N) \cup \operatorname{Ass}_S(M/N)$$

□

Local cohomology modules

Let $I = (f_1, \ldots, f_k)$ be an ideal of $S = K[z_1, \ldots, z_n]$, and M an S-module. The ith local cohomology module $H^i_I(M)$ is the $(k-i)$th homology of the Čhec complex:

$$0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j} \to \ldots \to M_{f_1 \cdots f_k} \to 0,$$

where the maps are the natural ones multiplied by a suitable sign. If each module of the above complex were a holonomic $\mathcal{D}$-module, then also the cohomology would be holonomic, since holonomicity is closed under short exact sequences. So, in this case, $H^i_I(M)$ would have a finite number of associated primes.

Our goal now is to show that S_f is a holonomic $\mathcal{D}$-module $\forall\ f \in S$.

Holonomic $\mathcal{D}$-modules

Recall that, if $\mathcal{G}$ is a good filtration on a left $\mathcal{D}$-module, then for any other filtration $\mathcal{H}$ on M, there is $r \in \mathbb{Z}$ such that, for all $t \in \mathbb{Z}$, $\mathcal{G}_t \subseteq \mathcal{H}_{t+r}$. In particular, $\forall\ \epsilon > 0\ \ \exists\ t_0$ such that:

$$(*) \qquad \dim_K(\mathcal{H}_t) \geq \frac{e(M) - \epsilon}{\dim(M)!} t^{\dim(M)} \qquad \forall t \geq t_0$$

LEMMA: Let M be a left $\mathcal{D}$-module with a filtration $\mathcal{G}$. If there is $c \in \mathbb{R}_{>0}$ such that $\dim_K(\mathcal{G}_t) \leq ct^n$ for $t \gg 0$, then M is holonomic. In particular, it is finitely generated!

Proof: Let M_0 be a finitely generated $\mathcal{D}$-submodule of M and set $\mathcal{G}'_t = \mathcal{G}_t \cap M_0\ \forall\ t \in \mathbb{Z}$, which defines a filtration on M_0. Then $\dim(M_0) \leq n$ and $e(M_0) \leq n!c$ by (*). Therefore M has ACC for finitely generated submodules. So it is finitely generated, and holonomic again by (*). $\square$

Holonomic $\mathcal{D}$-modules

THEOREM: Let M be a holonomic $\mathcal{D}$-module and $f \in S$. Then M_f is holonomic.

Proof: Let $\mathcal{G}$ be a good filtration on M, and δ the maximum degree of a monomial in the support of f. Define $\mathcal{H} = \{\mathcal{H}_t\}_t$ on M_f as:

$$\mathcal{H}_t = \left\langle \frac{m}{f^t} : m \in \mathcal{G}_{t(\delta+1)} \right\rangle.$$

That $\cup_t \mathcal{H}_t = M_f$ and $\mathcal{H}_t \subseteq \mathcal{H}_{t+1}$ is easy. If $m/f^t \in \mathcal{H}_t$, then obviously $x_i \cdot m/f^t \in \mathcal{H}_{t+1}$. Furthermore:

$$\partial_i \cdot \frac{m}{f^t} = \frac{f^t \cdot (\partial_i \cdot m) - tf^{t-1}\partial f/\partial z_i \cdot m}{f^{2t}} = \frac{f \cdot (\partial_i \cdot m) - t\partial f/\partial z_i \cdot m}{f^{t+1}} \in \mathcal{H}_{t+1}.$$

So $\mathcal{H}$ is a filtration of M_f such that:

$$\dim_K(\mathcal{H}_t) \leq \dim_K(\mathcal{G}_{t(1+\delta)}) \sim \frac{e(M)(1+\delta)^n}{n!} t^n$$

By the previous lemma M_f is holonomic. $\square$

Finiteness properties of $H_I^i(S)$

We saw last week that S has a structure of holonomic $\mathcal{D}$-module. So by the above theorem S_f is holonomic for all $f \in S$. If I is an ideal of S generated by $f_1, \ldots, f_k$ then, by meaning of the Čech complex, the local cohomology module $H_I^i(S)$ is a subquotient of:

$$\bigoplus_{1 \leq \ell_1 < \ldots < \ell_i \leq k} S_{f_{\ell_1} \cdots f_{\ell_i}}.$$

In particular $H_I^i(S)$ is a holonomic $\mathcal{D}$-module. As a consequence:

THEOREM: For any ideal $I \subseteq S$ the set $\mathrm{Ass}_S(H_I^i(S))$ is finite. More generally, $\mathrm{Ass}_S(H_I^i(M))$ is finite $\forall$ holonomic $\mathcal{D}$-modules M.

Finiteness properties of $H^i_I(S)$

THEOREM: Let $\mathfrak{m} = (z_1, \ldots, z_n) \subseteq S = K[z_1, \ldots, z_n]$.

(i) Then $H^n_{\mathfrak{m}}(S) \cong \mathcal{D}/\mathcal{D}\mathfrak{m}$ as left $\mathcal{D}$-modules. In particular, $\mathcal{D}/\mathcal{D}\mathfrak{m}$ is isomorphic to the injective hull $E_S(K)$ of $K = S/\mathfrak{m}$.

(ii) If M is an $\mathfrak{m}$-torsion $\mathcal{D}$-module, then $M \cong \bigoplus_{\lambda\in\Lambda} \mathcal{D}/\mathcal{D}\mathfrak{m}$ as left $\mathcal{D}$-modules. In particular M is an injective S-module.

Proof: (i) The map of $\mathcal{D}$-modules $\mathcal{D} \to H^n_{\mathfrak{m}}(S)$ sending 1 to $[1/z_1 \cdots z_n]$ is surjective, and one can check that its kernel is $\mathcal{D}\mathfrak{m}$.

(ii) Consider $\mathrm{soc}(M) \subseteq M$ and a K-basis $\{m_\lambda\}_\Lambda$ of $\mathrm{soc}(M)$. This gives rise to the following diagram of S-modules:

$$\begin{array}{ccc} \bigoplus_{\lambda\in\Lambda} \mathcal{D}/\mathcal{D}\mathfrak{m} & \xrightarrow{\;f\;} & M \\ \uparrow & & \uparrow \\ \bigoplus_{\lambda\in\Lambda} K & \xrightarrow{\;\cong\;} & \mathrm{soc}(M) \end{array}$$

The inclusion on the left is essential, so f is injective. Therefore $M \cong \bigoplus_{\lambda\in\Lambda} \mathcal{D}/\mathcal{D}\mathfrak{m} \bigoplus C$. Since $\mathrm{soc}(C) = 0$, we infer that $C = 0$. $\square$

Finiteness properties of $H^i_I(S)$

The above result implies that, if $H^i_I(S)$ is $\mathfrak{m}$-torsion, then $\exists\ s \in \mathbb{N}$:

$$H^i_I(S) \cong E_S(K)^s.$$

There are several interesting situations in which $H^i_I(S)$ is $\mathfrak{m}$-torsion, for example if $i > \mathrm{ht}(I)$ and I defines a smooth projective scheme. So s is an interesting number. Quite surprisingly, it is an invariant of S/I, we will soon discuss this aspect in more generality. First, let me say that, with not much more effort, we could prove:

THEOREM: $\mathrm{injdim}(H^i_I(S)) \leq \dim(\mathrm{Supp}(H^i_I(S)))$ $(\leq i)$.

Finiteness properties of $H_I^i(S)$

Recall that the *Bass numbers* of an S-module M are defined as:

$$\mu_i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})}(\mathrm{Ext}^i_{S_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})),$$

where $\mathfrak{p}$ is a prime ideal of S. Another way to think at them is the following: Every S-module M admits a minimal injective resolution

$$0 \to M \to E^0 \to E^1 \to E^2 \dots$$

Then $E^i \cong \bigoplus_{\mathfrak{p}\in \mathrm{Spec}(S)} E_S(S/\mathfrak{p})^{\mu_i(\mathfrak{p},M)}$.

THEOREM: A holonomic $\mathcal{D}$-module has finite Bass numbers. In particular, $\mu_i(\mathfrak{p}, H_I^j(S))$ is a finite number for all triples $\mathfrak{p}, i, j$.

The Lyubeznik numbers of a local ring containing a field

All the results stated for S hold true for any regular local ring R containing K. The point is that $\widehat{R} \cong K[[x_1, \ldots, x_n]]$. The algebra of differentials $D(K[[\underline{x}]], K)$ of $\widehat{R}$ is left-Noetherian and well described, so one can play a similar game to the previous one replacing $\mathcal{D}$ by $D(K[[\underline{x}]], K)$. Finally one can descend everything to R, essentially because $K[[\underline{x}]]$ is a faithfully flat R-algebra.
Besides all the previous beautiful results, Lyubeznik supplied us new invariants to play with:

DEFINITION-THEOREM: Let A be a local ring containing K. By Cohen-structure theorem we have a surjection $K[[x_1, \ldots, x_n]] \xrightarrow{\pi} \widehat{A}$. Denoting by $I = \mathrm{Ker}(\pi)$ and $\mathfrak{m}$ the maximal ideal of $K[[\underline{x}]]$, the finite numbers $\mu_p(\mathfrak{m}, H_I^{n-i}(K[[\underline{x}]])$ depend only on A, p and i. These invariants of A are usually denoted by $\lambda_{p,i}(A)$ and called the Lyubeznik's numbers of A.

Open problems

(i) The conjecture of Lyubeznik is still unsolved: $\mathrm{Ass}_R(H^i_I(R))$ are finite sets for any regular ring R.

(ii) A question of Lyubeznik: If A is a standard graded K-algebra with maximal ideal $\mathfrak{m}$, are $\lambda_{p,i}(A_\mathfrak{m})$ invariants of $\mathrm{Proj}(A)$? (Zhang, *Adv. Math.* 2011: Yes in positive characteristic).

(iii) One can show that $\lambda_{p,i} = 0$ if $i > d = \dim(A)$, $p > i$, or $p \geq i - 1$ and $i < \mathrm{depth}(A)$. Is $\lambda_{i-2,i}(A) = 0$ for all $i < \mathrm{depth}(A)$?

(iv) Compute the entire table $\lambda_{p,i}(A_\mathfrak{m})$ for determinantal rings A.