A SPECIAL FEATURE OF QUADRATIC MONOMIAL IDEALS

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The aim of the talk has been to show a property peculiar of quadratic monomial ideals, namely the Main Lemma below, proved in collaboration with Giulio Caviglia and Alexandru Constantinescu in [CCV]. The original motivation for looking up such property was to use it in the study of a conjecture by Gil Kalai, for which indeed it has been extremely helpful. During the talk, I mentioned how this lemma could be exploited in other situations. First of all let us recall it:

Main Lemma. *Let k be an infinite field, and $I \subseteq k[x_1, \ldots, x_n] =: S$ an ideal of height c. If I is monomial and generated in degree 2, then there exist linear forms $\ell_{i,j}$ for $i \in \{1, 2\}$ and $j \in \{1, \ldots, c\}$ such that $\ell_{1,1}\ell_{2,1}\ , \ \ldots \ , \ \ell_{1,c}\ell_{2,c}$ is an S-regular sequence contained in I.*

In general, if R is a standard graded Cohen-Macaulay k-algebra, where k is an infinite field, and $J \subseteq R$ is a height c homogeneous ideal generated in a single degree d, then we can always find an R-regular sequence $g_1, \ldots, g_c$ of degree d elements inside J. So, with the notation of the Main Lemma, since the polynomial ring S is Cohen-Macaulay, we already know that there is an S-regular sequence $f_1, \ldots, f_c$ consisting of quadratic polynomials inside I. The point of the result is that we can choose each f_i being a product of two linear forms.

Before sketching the (elementary) proof of the Main Lemma, let us see that the analog property fails for nonquadratic monomial ideals.

Example. Let J be the height 2 monomial ideal $(x^2y, y^2z, xz^2) \subseteq k[x, y, z]$. It is straightforward to check that the only products of linear forms $\ell_1\ell_2\ell_3$ in J are (scalar multiples of) the monomials x^2y, y^2z and z^2x. Clearly, no combination of 2 such monomials form a $k[x, y, z]$-regular sequence.

Proof of the Main Lemma. The proof goes like follows: Take a height c minimal prime $\mathfrak{p}$ of I

$$\mathfrak{p} = (x_1, \ldots, x_c)$$

(this of course can be done after relabeling the variables). We decompose the k-vector space generated by the quadratic monomials of I as:

$$I_2 = \bigoplus_{i=1}^{c} x_i V_i,$$

where $V_i = \langle x_j : x_i x_j \in I \ \text{ and } \ j \geq i \rangle$. We aim to find $\ell_i \in V_i$ such that

$$\dim_k(\langle x_i : i \in A\rangle + \langle \ell_i : i \in [c] \setminus A\rangle) = c \quad \forall\ A \subseteq [c] = \{1, \ldots, c\}.$$

Indeed, one can easily check that the condition above characterizes the fact that $x_1\ell_1, \ldots, x_c\ell_c$ is an S-regular sequence. The trick to find such linear forms ℓ_i is to construct a family of bipartite graphs G_A, for all $A \subseteq [c]$, in the following way:

(i) $V(G_A) = [c] \cup \{x_1, \ldots, x_n\}$;
(ii) $E(G_A) = \{\{i, x_i\} : i \in [c] \setminus A\} \cup \{\{i, x_j\} : i \in A \ \text{ and } \ x_j \in V_i\}$.

Claim: There is a matching of G_A containing all the vertices of $[c]$. To show this, we will appeal to the Marriage Theorem, by showing that, for a subset $B \subseteq [c]$, the set $N(B)$ of vertices adjacent to some vertex in B has cardinality not smaller than that of B. To this purpose, note that:

$$|N(B)| = \dim_k\Big(\sum_{i\in A\cap B} V_i + \sum_{i\in([c]\setminus A)\cap B}\langle x_i\rangle\Big) =$$

$$\dim_k\Big(\sum_{i\in A\cap B} V_i + \sum_{i\in[c]\setminus(A\cap B)}\langle x_i\rangle\Big) - \dim_k\Big(\sum_{i\in[c]\setminus B}\langle x_i\rangle\Big) \geq |B|,$$

where the inequality follows from the fact that the ideal $\sum_{i\in A\cap B}(V_i) + (x_i : i \in [c]\setminus(A\cap B))$, containing I, must have height at least c. Therefore, we get by the Marriage Theorem the existence of

$$j(1,A),\ j(2,A), \ldots,\ j(c,A)$$

such that $\{1, x_{j(1,A)}\}, \ldots,\ \{c, x_{j(c,A)}\}$ is a matching of G_A, which implies that

$$\dim_k\langle x_{j(1,A)}, \ldots, x_{j(c,A)}\rangle = c.$$

Now it is enough to put

$$\ell_i = \sum_{A\subseteq[c]} \lambda(A) x_{j(i,A)},$$

where $\lambda(A)$ are general elements of k. □

As already mentioned, we were motivated in proving the Main Lemma to study the following conjecture of Kalai:

Conjecture. *(Kalai): The f-vector of a Cohen-Macaulay flag simplicial complex is the f-vector of a Cohen-Macaulay balanced simplicial complex.*

It is convenient to recall here that the f-vector $(f_{-1}, \ldots, f_{d-1})$ of a $(d-1)$-dimensional simplicial complex Δ is defined as:

$$f_j = |\{j\text{-dimensional faces of } \Delta\}|$$

As a consequence of the Main Lemma, we get the following:

Theorem. *The h-vector of a Cohen-Macaulay flag simplicial complex is the h-vector of a Cohen-Macaulay balanced simplicial complex.*

To show how the theorem above follows form the Main Lemma, we remind that, if $(h_0, \ldots, h_d)$ and $(f_{-1}, \ldots, f_{d-1})$ are, respectively, the h- and f-vector of a given $(d-1)$-dimensional simplicial complex, then the following equations hold true:

$$h_j = \sum_{i=0}^{j}(-1)^{j-i}\binom{d-i}{j-i} f_{i-1} \quad \text{and} \quad f_{j-1} = \sum_{i=0}^{j}\binom{d-i}{j-i} h_i$$

From these formulas one sees that the theorem above is very close to answering positively the conjecture of Kalai: The only defect is that h- and f-vector determine each other only if the dimension of the simplicial complex is known, and the h-vector does not provide such an information (in fact it may happen that $h_d = 0$).

For the convenience of the reader, let us recall the definitions of the objects occurring in the conjecture of Kalai:

(i) A simplicial complex Δ is said to be Cohen-Macaulay (over k) if its Stanley-Reisner ring $k[\Delta]$ is Cohen-Macaulay;
(ii) A simplicial complex is said to be flag if all its minimal nonfaces have cardinality 2;
(iii) A $(d-1)$-dimensional simplicial complex is said to be balanced if its 1-skeleton is a d-colorable graph.

Why were we able to prove the "h-version" of the conjecture of Kalai and not the original one? The point is that the entries of the h-vector of a simplicial complex Δ are the coefficients of the h-polynomial of $k[\Delta]$. So, by combining the Main Lemma with the main result obtained by Abedelfatah in [Ab1], we get the following:

Theorem. A1. *The h-vector of a $(d-1)$-dimensional Cohen-Macaulay flag simplicial complex on $n+d$ vertices equals to the Hilbert function of S/J where J is an ideal of S containing $(x_1^2,\ldots,x_n^2)$.*

Notice that the ideal J of Theorem A1 can be chosen monomial (just passing to the initial ideal). In this case, the Hilbert function of S/J is just the f-vector of the simplicial complex Γ on n vertices whose faces are $\{i_1,\ldots,i_r\}$ where $x_{i_1}\cdots x_{i_r}\notin J$. Therefore Theorem A1 can be re-stated as:

Theorem. A2. *The h-vector of a $(d-1)$-dimensional Cohen-Macaulay flag simplicial complex Δ on $n+d$ vertices is the f-vector of some simplicial complex Γ on n vertices.*

On the other hand, Theorem A2 is in turn equivalent to:

Theorem. A3. *The h-vector of a $(d-1)$-dimensional Cohen-Macaulay flag simplicial complex Δ on $n+d$ vertices is the h-vector of an $(n-1)$-dimensional Cohen-Macaulay balanced simplicial complex Ω on $2n$ vertices.*

Proof. By Theorem A2 we know that there is a simplicial complex Γ on n vertices with f-vector equal to the h-vector of Δ. Set:

$$J = I_\Gamma + (x_1^2,\ldots,x_n^2) \subseteq S$$

and consider the polarization J' of J in $S[y_1,\ldots,y_n]$. Then $J' = I_\Omega$ where, since the polarization preserves the minimal graded free resolution, the h-vector of Ω is the Hilbert function of S/J (which is the f-vector of Γ) and Ω is Cohen-Macaulay.

Now, notice that Ω is $(n-1)$-dimensional, and the coloring $\mathrm{col}(x_i) = \mathrm{col}(y_i) = i$ provides an n-coloring of the 1-skeleton of Ω, so that Ω is balanced. □

Remark. To show that Theorem A3 $\implies$ Theorem A1, one has to use that the Stanley-Reisner ring of a pure $(d-1)$-dimensional balanced simplicial complex Ω has a special system of parameters, namely:

$$\ell_i = \sum_{\mathrm{col}(x_j)=i} x_j \quad \forall\, i = 1,\ldots,d.$$

If Ω is on m vertices, then $k[\Omega]/(\ell_1,\ldots,\ell_d) \cong k[x_1,\ldots,x_{m-d}]/J$, where J is an ideal containing the squares of the variables. Furthermore, if Ω is Cohen-Macaulay, then $\ell_1,\ldots,\ell_d$ is a $k[\Omega]$-regular sequence, so the h-polynomial of $k[\Omega]/(\ell_1,\ldots,\ell_d)$ is the same as the h-polynomial of $k[\Omega]$.

Remark. Theorem A1 can be seen as the solution of a particular case of a general conjecture of Eisenbud-Green-Harris, namely the *quadratic monomial case*. For the precise statement of the conjecture see [EGH1, EGH2]. During the conference in Cortona, Abedelfatah posted on the arxiv a solution of the EGH conjecture in the *monomial case* (any degree), see [Ab2]. His proof is not an obvious extension of ours, essentially because the Example given in the first page.

At the end of the talk, I discussed another aspect in which the Main Lemma might be helpful: The dual graph $G(\Delta)$ of a pure $(d-1)$-dimensional simplicial complex Δ is defined as:

(i) $V(G(\Delta)) = \{\text{facets of } \Delta\}$;
(ii) $E(G(\Delta)) = \{\{F,G\} : |F\cap G| = d-1\}$.

Recently, Adiprasito and Benedetti proved in [AB] that, if Δ is a $(d-1)$-dimensional Cohen-Macaulay flag simplicial complex on n vertices, then

$$\mathrm{diam}(G(\Delta)) \leq n-d.$$

The Main Lemma might be helpful to get further understanding of $G(\Delta)$ for Δ flag as follows: Recall that

$$I_\Delta = \bigcap_{F \text{ facet of } \Delta} (x_i : i \notin F),$$

so the graph $G(\Delta)$ may be thought also as the graph of minimal primes of I_Δ; i.e. the graph whose vertices are the minimal primes of I_Δ, and two minimal primes are connected by an edge if and only if the height of their sum is one more than the height of I_Δ. The Main Lemma says that there exist linear forms $\ell_{i,j}$ such that $\ell_{1,1}\ell_{2,1}, \ldots, \ell_{1,n-d}\ell_{2,n-d}$ is an S-regular sequence contained in I_Δ. This implies that the graph $G(\Delta)$ is an induced subgraph of the graph of minimal primes of the ideal

$$(\ell_{1,1}\ell_{2,1}, \ldots, \ell_{1,n-d}\ell_{2,n-d}).$$

Such a graph is quite simple to describe: It is obtained by contracting some edges (which ones depends on the geometry of the matroid given by $\ell_{1,1}, \ell_{2,1}, \ldots, \ell_{1,n-d}, \ell_{2,n-d}$) of the graph $\mathbb{G}$ defined as:

- $V(\mathbb{G}) = 2^{\{1,\ldots,n-d\}}$;
- $\{A, B\} \in E(\mathbb{G})$ if and only if $|A \cup B| - |A \cap B| = 1$.

This should give strong restrictions on the structure of $G(\Delta)$ for Δ flag.

References

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