ON THE DUAL GRAPHS OF COMPLETE INTERSECTIONS

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- By the Nullstellensatz, we have $\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}$.

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An algebraic variety $X \subseteq \mathbb{P}^n$ of codimension c is a complete intersection (CI) if $\mathcal{I}(X)$ is generated by c polynomials.

An algebraic variety $X \subseteq \mathbb{P}^n$ is called a subspace arrangement if it is the union of linear subspaces of $\mathbb{P}^n$.

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We are going to inquire on the connectedness properties of $G(C)$ given global properties of C.

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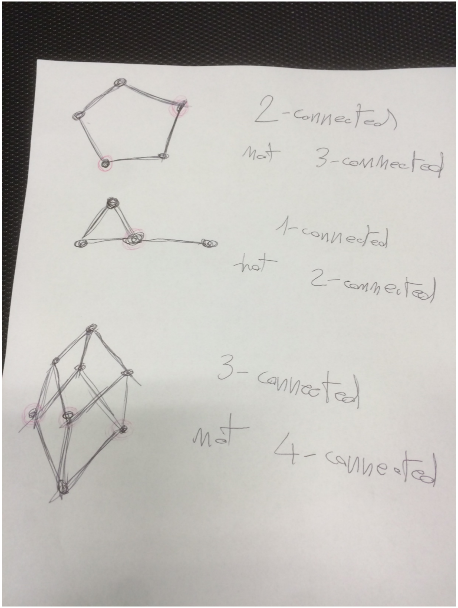
A graph G is r-connected if it has at least r vertices and removing $< r$ vertices yields a connected graph. In particular:

- G is connected $\Leftrightarrow$ G is 1-connected;
- G is $(r+1)$-connected $\Rightarrow$ G is r-connected.

Examples of r-connectivity



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For example, if the ideal of definition of C is defined by 2 cubics, then $G(C)$ will be 4-connected.

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$$\ell_{11}, \ldots, \ell_{1d}, \ell_{21}, \ldots, \ell_{2e} \in S = K[x_0, x_1, x_2, , x_3]$$

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In this case $f = \ell_{11} \cdots \ell_{1d}$ and $g = \ell_{21} \cdots \ell_{2e}$ will do the job. If the ℓ_{ij}'s are general enough, precisely if each four of them are linearly independent, then one can check that the dual graph of C is not $(d + e - 1)$-connected.

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And what about the possible graphs arising as dual graphs of CI line arrangement? One can see that not any graph arises as the dual graph of a (not necessarily CI) line arrangement. On the other hand, recently Kollar proved that any graph is the dual graph of a projective curve!

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A curve $C \subseteq \mathbb{P}^3$ which is the intersection of two surfaces is called a set-theoretic complete intersection (SCI).

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Notice that C is connected if and only if the dual graph $G(C)$ is connected. Therefore the above result implies that is plenty of line arrangements which are SCI without being a CI .....

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By construction, the dual graph $G(C)$ is connected, so C is a set-theoretic complete intersection. However, $G(C)$ is not even 2-connected, whereas N can be arbitrarily large ...

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If X is a complete intersection, then it is arithmetically Gorenstein. Furthermore, the Castelnuovo-Mumford regularity of $S/\mathcal{I}(X)$ is

$$\sum_{i=1}^{c} \deg(f_i) - c$$

where $\mathcal{I}(X) = (f_1, \ldots, f_c)$ and $c = \operatorname{codim}_{\mathbb{P}^n} X$.

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where $L \subseteq \mathbb{P}^n$ is a general linear space of dimension c.

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For example, if X is a linear space then its degree is 1.

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This recovers the result for subspace arrangements, since each irreducible component, being a linear space, has degree 1.

Actually we proved more: namely, we showed a version of the previous theorem for any arithmetically Gorenstein projective scheme (not necessarily reduced). For such a version, one needs that d bounds from above the Castelnuovo-Mumford regularity of each component.

THANK YOU !!