J-MULTIPLICITY OF DETERMINANTAL IDEALS

Joint work with Jack Jeffries and Jonathan Montaño

Preliminaries

Let $(R, \mathfrak{m})$ be a d-dimensional local ring with $|R/\mathfrak{m}| = \infty$. Given an ideal $I \subseteq R$, we form the associated graded ring:

$$G = \mathrm{gr}_I(R) = \bigoplus_{k \in \mathbb{N}} I^k / I^{k+1}$$

(here the direct sum is taken as R-modules). It turns out that G is a d-dimensional standard graded algebra over R/I. Let $\mathfrak{m}G$ be the extension of $\mathfrak{m}$ to G, and introduce the fiber cone:

$$F = F(I) = G/\mathfrak{m}G = \bigoplus_{k \in \mathbb{N}} I^k / \mathfrak{m} I^k.$$

F is a standard graded $R/\mathfrak{m}$-algebra. Its dimension is known as the analytic spread of I and denoted by $\ell(I)$.

Preliminaries

Notice that, for all $i \in \mathbb{N}$, we have:

$$H^i_{\mathfrak{m}G}(G) \cong H^i_{\mathfrak{m}}(G) \cong \bigoplus_{k \in \mathbb{N}} H^i_{\mathfrak{m}}(I^k/I^{k+1}).$$

So, by $0 \to I^k/I^{k+1} \to R/I^{k+1} \to R/I^k \to 0$ we get:

$$\mathrm{grade}(\mathfrak{m}G, G) = \min_k\{\mathrm{depth}(I^k/I^{k+1})\} = \min_k\{\mathrm{depth}(R/I^k)\}.$$

In particular $\ell(I) \leq d - \min_k\{\mathrm{depth}(R/I^k)\}$ (Burch), with equality holding if G is Cohen-Macaulay (Eisenbud-Huneke).

Definition of j-multiplicity

Let us analyze the G-submodule $M = H^0_{\mathfrak{m}G}(G) \subseteq G$:

(i) Because both $\mathfrak{m}G$ and G are graded, then M is a graded G-submodule of G;

(ii) By Noetherianity there is $N \gg 0$ for which $(\mathfrak{m}G)^N M = 0$, therefore M is actually a finitely generated graded $G/(\mathfrak{m}G)^N$-module;

(iii) Since $G/(\mathfrak{m}G)^N$ is a standard graded algebra over $R/\mathfrak{m}^N$ (that is local Artinian), the function $n \mapsto \dim_{R/\mathfrak{m}} M_n$ is eventually a polynomial $P(n)$;

(iv) The above function is a multiple of $n \mapsto \text{length}_{R/\mathfrak{m}^N} M_n$, so $P(n)$ has degree $\dim(M) - 1 \leq \dim(G/\mathfrak{m}G) - 1 = \ell(I) - 1$.

DEFINITION: The j-multiplicity of I is the natural number:

$$j(I) = \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} P(n) = \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \dim_{R/\mathfrak{m}} H^0_{\mathfrak{m}}(I^n/I^{n+1})$$

Fiddling with j-multiplicity

Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be a standard graded ring over an infinite field $R_0 = K$, $\mathfrak{m} = \bigoplus_{n>0} R_n$ the maximal irrelevant ideal and $I \subseteq R$ is a graded ideal. Everything said before holds in this situation by letting the maximal irrelevant ideal play the role of the former unique maximal ideal. In particular

$$j(I) = \lim_{n \to \infty} \frac{(d-1)!}{n^{d-1}} \dim_K H^0_{\mathfrak{m}}(I^n/I^{n+1})$$

REMARK: There are ideals I such that $H^0_{\mathfrak{m}}(I^n/I^{n+1}) = 0$ for all n, for example any complete intersection of height $< d$ in a regular R: For such ideals I in fact $\operatorname{depth}(I^n/I^{n+1}) > 0$ for all $n \in \mathbb{N}$.

Fiddling with j-multiplicity

Example: Let us consider

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \end{pmatrix}$$

$R = K[X]$ and $I = I_2(X)$. The fiber cone of I is the coordinate ring of the Grassmannian $G(2,6)$. In particular its dimension is 9, so $\ell(I) = 9 < 12 = \dim(R)$. One can show that the associated graded ring is Cohen-Macaulay, so by Eisenbud-Huneke we infer

$$\min_k \{\mathrm{depth}(I^k/I^{k+1})\} = 12 - 9 = 3.$$

In particular $H^0_{\mathfrak{m}}(I^n/I^{n+1}) = 0$ for all n, so $j(I) = 0$.

Basic results

There are examples of ideals $I \subseteq R$ with $\ell(I) < d = \dim(R)$ but nevertheless $H^0_{\mathfrak{m}}(G) \neq 0$. However we have a nice characterization for the vanishing of the j-multiplicity:

Proposition: The following are equivalent:

(i) $j(I) \neq 0$;

(ii) $\dim H^0_{\mathfrak{m}}(G) = d$.

(iii) $\ell(I) = d$.

Proof: Not obvious only $(iii) \Rightarrow (ii)$. Pick $\mathfrak{p} \in \mathrm{Supp}_G(F)$ such that $\dim G/\mathfrak{p} = d$. Then $\mathfrak{p} \in \mathrm{Min}_G(G) \subseteq \mathrm{Ass}_G(G)$, so there is a nonzero $x \in G$ such that $\mathfrak{p} = 0 :_G x$. Because $\mathfrak{m}G \subseteq \mathfrak{p}$, we have $\mathfrak{m}G \cdot x = 0$. Hence $x \in H^0_{\mathfrak{m}G}(G) \subseteq G$, so $\mathfrak{p} \in \mathrm{Ass}(H^0_{\mathfrak{m}G}(G))$. We therefore infer $\dim H^0_{\mathfrak{m}G}(G) = d$.

Basic results

As I should have already mentioned, the j-multiplicity of an $\mathfrak{m}$-primary ideal I agrees with the Hilbert-Samuel multiplicity of I. In fact the j-multiplicity was introduced by Achilles-Manaresi to extend the good features of Hilbert-Samuel multiplicity to the non-$\mathfrak{m}$-primary situation. For example, Flenner-Manaresi proved:

$$f \in \bar{I} \Leftrightarrow j(I_{\mathfrak{p}}) = j((I + (f))_{\mathfrak{p}}) \ \forall \ \mathfrak{p} \in \mathrm{Spec}(R)$$

However to compute the j-multiplicity seems a really difficult problem. So far one of the few successful ways to get it is provided by a length formula of Achilles-Manaresi: If $a_1, \ldots, a_d$ are general elements of I, then

$$j(I) = \mathrm{length}_{R/\mathfrak{m}}\left(\frac{R}{(a_1, \ldots, a_{d-1}) : I^{\infty} + (a_d)}\right)$$

Known examples

Various generalizations and applications of the previous formula were given by Nishida-Ulrich, who in particular were able to show that $j(I) = 4$ where I is the ideal of $R = K[x_1, \dots, x_5]$ generated by the 2-minors of the matrix

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}$$

A large class for which the j-multiplicity is known is provided by a recent result of Jeffries-Montaño, who express the j-multiplicity of any monomial ideal $I \subseteq R = K[x_1, \dots, x_d]$ as the volume of a polytopal complex in $\mathbb{R}^d$ described by the exponents of the minimal monomial generators of I.

Our contribute

With Jeffries and Montaño we express the j-multiplicities of the ideal generated by the t-minors of a generic $m \times n$-matrix (respectively t-minors of a generic symmetric $n \times n$-matrix) (respectively $2t$-pfaffians of a generic alternating $n \times n$-matrix) as an interesting integral in $\mathbb{R}^m$. Actually we are able to express it also as the volume of a polytope in $\mathbb{R}^{mn}$, but the above integrals are tantalizing related to certain quantities in random matrix theory!

We also give a combinatorial formula for the j-multiplicity of the ideal generated by the t-minors of a Hankel matrix (the kind of matrix of the Nishida-Ulrich example).

$m \times n$-generic matrices

Let's discuss the case of generic $m \times n$-matrices. So

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & \cdots & x_{mn} \end{pmatrix}.$$

$R = K[X]$ and $I = I_t(X)$ is the ideal generated by the t-minors of X. Without loss of generality, we assume $t \le m \le n$. If $t = m$ then the fiber cone of I is the coordinate ring of the Grassmannian $G(m, n)$, So $\ell(I) = m(n-m) + 1 < mn = \dim(R)$ in this case, therefore $j(I_m(X)) = 0$. We will thus place ourselves in the case $t < m$, where $\ell(I) = mn$.

2-minors of a 3×3

So the first nontrivial case is $t = 2$, $m = n = 3$. In the remaining part of the talk we will carry on this example, i. e. I is the ideal of 2-minors of a 3×3-matrix, and R is a polynomial ring in 9 variables over a field K. By a result of Bruns

$$I^s = \mathfrak{m}^{2s} \cap I^{(s)}.$$

Because

$$H^0_{\mathfrak{m}}(I^s/I^{s+1}) = \frac{(I^{s+1})^{sat} \cap I^s}{I^{s+1}} = \frac{(I^{(s+1)}) \cap I^s}{I^{s+1}},$$

we wish to compute the dimension of $\frac{(I^{(s+1)}) \cap I^s}{I^{s+1}}$.

2-minors of a 3×3

To this aim we have to consider the K-basis of $R = K[X]$ consisting of standard monomials, i.e. products of minors of X forming an ascending chain with respect to a certain partial order on the minors of X. For example:

$$[123|123]^2 \cdot [13|13] \cdot [23|13]^4 \cdot [2|1]$$

is a standard monomial. Instead the following

$$[123|123]^2 \cdot [13|23] \cdot [23|13]^4 \cdot [2|1]$$

is not.

For a product of minors $\Delta = \delta_1 \cdots \delta_k$, where δ_i is an a_i-minor, the vector $(a_1, \ldots, a_k)$ is referred to be the shape of Δ. If Δ is a standard monomial, then $a_1 \geq \ldots \geq a_k$.

2-minors of a 3×3

Quite surprisingly, whether a product of minors $\Delta = \delta_1 \cdots \delta_k$ of shape $(a_1, \dots, a_k)$ belongs or not to $I^{(s)}$ depends only on its shape:

$$\Delta \in I^{(s)} \Leftrightarrow \sum_{i=1}^{k} (a_i - 1) \geq s$$

For what we said till now the K-dimension of $H^0_{\mathfrak{m}}(I^s / I^{s+1})$ is given by the of standard monomials in the set $A(s) = (I^{(s+1)} \cap I^s) \setminus I^{s+1}$. Thanks to the mentioned results, we can infer that a shape $(a_1, \dots, a_k)$ occurs in $A(s)$ iff one of the following two holds:

$$\begin{cases} 2s - 2x - y = 3|\{i : a_i = 3\}| \\ x + 2y \leq s - 3 \\ x, y \geq 0 \end{cases} \qquad \begin{cases} 2s - 2x - y + 1 = 3|\{i : a_i = 3\}| \\ x + 2y \leq s - 1 \\ x, y \geq 0 \end{cases}$$

where $x = |\{i : a_i = 1\}|$ and $y = |\{i : a_i = 2\}|$.

2-minors of a 3×3

We now have to count how many standard monomials are there of a given shape $(a_1, \ldots, a_k)$. After a careful manipulation of the hook length formula we get that this number is:

$$1/4((x+1)+(y+1))^2(x+1)^2(y+1)^2$$

where $x = |\{i : a_i = 1\}|$ and $y = |\{i : a_i = 2\}|$. So the dimension as a K-vector space of $H^0_{\mathfrak{m}}(I^s/I^{s+1})$, call it $j(s)$, is about:

$$\frac{1}{3}\Big(\frac{1}{4}\sum_{\substack{(x,y)\in\mathbb{N}^2\\ x+2y\leq s-3}}\big((x+1)+(y+1)\big)^2(x+1)^2(y+1)^2$$

$$+\frac{1}{4}\sum_{\substack{(x,y)\in\mathbb{N}^2\\ x+2y\leq s-1}}\big((x+1)+(y+1)\big)^2(x+1)^2(y+1)^2\Big)$$

Riemann sums

Let us call $f(x,y) = (x+y)^2 x^2 y^2$ and $\mathcal{R}$ the triangle of $\mathbb{R}^2$ $\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0 \text{ and } x + 2y \leq 1\}$. Then

$$j(s) \approx s^6/6 \sum_{(x,y)\in(1/s)\mathbb{N}^2\cap\mathcal{R}} f(x,y)$$

and by standard integration theory,

$$\frac{1}{s^2} \sum_{(x,y)\in(1/s)\mathbb{N}^2\cap\mathcal{R}} f(x,y) \approx \int_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Consequently,

$$j(s) \approx \frac{s^8}{6} \int_{\mathcal{R}} x^2 y^2 (x+y)^2 \,\mathrm{d}x\,\mathrm{d}y = \frac{s^8}{20160}\,.$$

We conclude that $j(I) = 8!/20160 = \boxed{2}$.

The general statement

Our general result is that, if I is the ideal generated by the t-minors of a generic $m \times n$-matrix, then:

$$j(I) = \frac{c}{m!} \int_{\substack{[0,1]^m \\ \sum_{i=1}^m x_i = t}} (x_1 \cdots x_m)^{n-m} \cdot \prod_{i<j} (x_i - x_j)^2 \, \mathrm{d}\sigma,$$

where $c = \dfrac{(nm-1)! \cdot t}{(n-1)!(n-2)! \cdots (n-m)! \cdot (m-1)! \cdots 1!}$.

Surprisingly, the exact evaluation of the above integral would give the probability for a $m \times m$ random Hermitian matrix Z with both Z and $Id - Z$ positive definite, with probability density function proportional to $\det(Z)^{n-m}$, to have trace $= t$. This seems to be an important problem in random matrix theory.