# Cohomological dimension of open subsets of the projective space

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# ZARISKI VS ÉTALE TOPOLOGY

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The most used lower bounds, setting $U = \mathbb{P}^n \setminus X$, come from:

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Let $U$ be a scheme over a field $K$. Its cohomological dimension is

$$\mathrm{cd}(U) = \sup\{i \in \mathbb{N} : \exists \text{ quasi-coherent sheaf } \mathcal{F} : H^i(U, \mathcal{F}) \neq 0\}.$$

The étale cohomological dimension of $U$ is

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We have:

1. $r \geq \text{cd}(U) + 1$.
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## Some results around the issue

The expectation of Hartshorne was not correct:

(Ogus, 1973): $\text{char}(K) = 0$, $X = v_s(\mathbb{P}^d) \subset \mathbb{P}^n$, $U = \mathbb{P}^n \setminus X$.

$$\text{écd}(U) = 2n - 2, \quad \text{cd}(U) = n - d - 1$$

In positive characteristic, negative answers can be produced using:

(Peskine-Szpiro, 1973): $\text{char}(K) > 0$, $X \subset \mathbb{P}^n$ $d$-dimensional arithmetically Cohen-Macaulay scheme, $U = \mathbb{P}^n \setminus X$. Then

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$X \subset \mathbb{P}^n$ smooth, $U = \mathbb{P}^n \setminus X$.

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The above result fails in characteristic 0.

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The above argument implies an interesting fact: If $I \subset S$ is a graded ideal such that $\mathrm{Proj}(S/\sqrt{I})$ is smooth over $K$ then:

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A nice application is a generalization of a result of Singh and Walther (2005): If $I$ defines $C \times X$, where $C$ is a smooth projective curve of positive genus and $X$ is any positive dimensional smooth projective scheme, then:

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