LYUBEZNIK NUMBERS AND DEPTH

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Bass numbers

Let R be a noetherian ring and M an R-module. Consider a minimal injective resolution:

$$0 \to M \to E^0 \to E^1 \to E^2 \to \dots$$

The indecomposable injective R-modules are $E_R(R/\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec} R$. The Bass numbers of M are defined as the number $\mu_i(\mathfrak{p}, M)$ of copies of $E_R(R/\mathfrak{p})$ occurring in E^i. In other words:

$$E^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec}(R)} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)}.$$

It turns out that $\mu_i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$.

In particular, if M is finitely generated, each Bass number is finite.

Cohomology with support

Throughout the talk all the rings and schemes we consider are noetherian.

Given a closed subset Y of a regular n-dimensional scheme $X = \mathrm{Spec}(S)$, we will freely use the following facts on the S-modules $H^i_Y(X, \mathcal{O}_X)$ (which may be not finitely generated):

(i) (Grothendieck) $\Rightarrow$ $H^i_Y(X, \mathcal{O}_X) = 0$ if $i > \dim(X)$ and if $i < \mathrm{codim}_X Y$. Also, given any S-module M and $\mathfrak{m} \subseteq S$:

$$H^i_{\mathfrak{m}}(M) = 0 \;\; \forall\; i > \dim(\mathrm{Supp}(M)).$$

(ii) (Hartshorne-Lichtenbaum) $\Rightarrow$ $H^n_Y(X, \mathcal{O}_X) = 0 \Leftrightarrow \dim(Y) > 0$

(iii) (Peskine-Szpiro, Ogus) $\Rightarrow$ If S is local, contains a field and $\mathrm{depth}(\mathcal{O}_Y(Y)) \geq 2$, then

$$H^{n-1}_Y(X, \mathcal{O}_X) = H^n_Y(X, \mathcal{O}_X) = 0$$

Lyubeznik numbers

Theorem (Huneke-Sharp, Lyubeznik): If S is a regular local ring containing a field, then each Bass number of $H^i_Y(X, \mathcal{O}_X)$ is finite for any closed subset $Y \subseteq X = \mathrm{Spec}(S)$ and all $i \in \mathbb{N}$.

Definition-Theorem (Lyubeznik): R local containing a field. The completion $\widehat{R}$ is isomorphic to S/I, where $I \subseteq S = k[[x_1, \ldots, x_n]]$. The Bass numbers $\mu_i(\mathfrak{m}, H^{n-j}_Y(X, \mathcal{O}_X))$, where $\mathfrak{m} = (x_1, \ldots, x_n)$ and $Y = \mathcal{V}(I) \subseteq \mathrm{Spec}(S) = X$, depend only on $Z = \mathrm{Spec}(R)$, i and j. The Lyubeznik numbers of Z are therefore defined as:

$$\lambda_{i,j}(Z) = \mu_i(\mathfrak{m}, H^{n-j}_Y(X, \mathcal{O}_X)).$$

He also showed that $H^i_{\mathfrak{m}}(H^{n-j}_Y(X, \mathcal{O}_X)) \cong E_S(k)^{\lambda_{i,j}(Z)}$.

Basic properties

For a while, R will be a local ring containing a field k,
$S = k[[x_1, \dots, x_n]]$, $\mathfrak{m} = (x_1, \dots, x_n)$ and $I \subseteq S$ s. t. $\widehat{R} \cong S/I$,
$X = \mathrm{Spec}(S)$, $Y = \mathcal{V}(I) \subseteq X$ and $Z = \mathrm{Spec}(R)$.

If $\dim(Z) = d$, then $\mathrm{codim}_X Y = n - d$. In particular, if $j > d$, $H_Y^{n-j}(X, \mathcal{O}_X)$ vanishes, therefore:

$$\lambda_{i,j}(Z) = 0 \qquad \forall\, j > d.$$

If the closure of $\mathfrak{p} \in X$ has dimension bigger than j, then

$$H_Y^{n-j}(X, \mathcal{O}_X)_{\mathfrak{p}} = H^{n-j}_{\mathrm{Spec}(\mathcal{O}_{Y,\mathfrak{p}})}(\mathrm{Spec}(S_{\mathfrak{p}}), \mathcal{O}_{\mathrm{Spec}(S_{\mathfrak{p}})}) = 0.$$

So $\dim \mathrm{Supp}(H_Y^{n-j}(X, \mathcal{O}_X)) \le j$. In particular, $H^i_{\mathfrak{m}}(H_Y^{n-j}(X, \mathcal{O}_X))$ vanishes whenever $i > j$, so that:

$$\lambda_{i,j}(Z) = 0 \qquad \forall\, i > j.$$

The Lyubeznik table

Thus the following $(d+1) \times (d+1)$ upper triangular matrix is an invariant of a d-dimensional affine scheme Z as above:

$$\Lambda(Z) = \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} & \lambda_{0,2} & \cdots & \lambda_{0,d} \\ 0 & \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,d} \\ 0 & 0 & \lambda_{2,2} & \cdots & \lambda_{2,d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_{d,d} \end{pmatrix}$$

.

(here $\lambda_{i,j} = \lambda_{i,j}(Z)$). The above matrix is pretty mysterious, however there are various results describing some of the entries...

Easy statements

(i) If all the irreducible components of Y have dimension at least b, then:

$$\lambda_{i,i}(Z) = 0 \quad \forall\ i < b.$$

Since $\mathcal{O}_{Y,\mathfrak{p}}$ has positive dimension for any $\mathfrak{p} \in X$ whose closure has dimension i, we have $\dim \mathrm{Supp}(H_Y^{n-i}(X, \mathcal{O}_X)) < i$ for all $i < b$.

(ii) If Y is a complete intersection, then $H_Y^{n-j}(X, \mathcal{O}_X) = 0$ for all $j < d$ because $X \setminus Y$ is covered by $n - d$ affines. So $\lambda_{i,j}(Z) = 0$ if $j < d$. Furthermore, because the second page of the spectral sequence

$$E_2^{i,j} = H_{\mathfrak{m}}^i(H_Y^{n-j}(X, \mathcal{O}_X))) \Rightarrow H_{\mathfrak{m}}^{n+i-j}(S)$$

is full of zeroes, it is easy to infer that $\lambda_{i,d}(Z) = \delta_{i,d}$.

More serious results

Theorem (Zhang): $\lambda_{d,d}(Z)$ is the number of connected components of the codimension 1 graph of $Y \times_k \overline{k}$.

Theorem (Blickle-Bondu): If $\mathcal{O}_{Y,\mathfrak{p}}$ is a complete intersection for any nonclosed point $\mathfrak{p} \in Y$, then $\lambda_{i,d}(Z) - \delta_{i,d} = \lambda_{0,d-i+1}(Z)$ and $\lambda_{i,j}(Z)$ vanishes whenever $0 < i$ and $j < d$.

Theorem (Garcia Lopez-Sabbah, Blickle-Bondu, Blickle): If, besides satisfying the condition above, $R = \mathcal{O}_{V,x}$ for a closed k-subvariety V of a smooth variety, then

$$\lambda_{0,j}(Z) = \begin{cases} \dim_{\mathbb{C}} H^j_{\{x\}}(V_{\mathrm{an}}, \mathbb{C}) & \text{if } k = \mathbb{C} \\ \dim_{\mathbb{Z}/p\mathbb{Z}} H^j_{\{x\}}(V_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) & \text{if } k = \mathbb{Z}/p\mathbb{Z} \end{cases}$$

Projective invariant?

Conjecture (Lyubeznik): Let X be a projective scheme over k. The Lyubeznik table of the spectrum of the coordinate ring of X (localized at the maximal irrelevant) is actually an invariant of X.

All the previous results provide evidence for the above conjecture.

(Zhang): True in positive characteristic!

Vanishing of $\lambda_{i,j}$ from the depth

Proposition: $\lambda_{i,j}(Z) = 0$ for all $j < \mathrm{depth}(R)$ and $i \geq j-1$.

Proof: If we pick $\mathfrak{p} \in Y$ such that $\dim(\mathcal{V}(\mathfrak{p})) = j-1$, then:

$$\mathrm{depth}(\mathcal{O}_{Y,\mathfrak{p}}) \geq 2,$$

which thereby implies

$$H^{n-j}_Y(X, \mathcal{O}_X)_{\mathfrak{p}} \cong H^{n-j}_{\mathrm{Spec}(\mathcal{O}_{Y,\mathfrak{p}})}(\mathrm{Spec}(S_{\mathfrak{p}}), \mathcal{O}_{\mathrm{Spec}(S_{\mathfrak{p}})}) = 0$$

so that $\dim \mathrm{Supp}(H^{n-j}_Y(X, \mathcal{O}_X)) < j-1$. $\square$

Vanishing of $\lambda_{i,j}$ from the depth

Notice that, if $\mathsf{char}(k) > 0$, then $\lambda_{i,j}(Z) = 0$ for all $j < \mathrm{depth}(R)$.

Proof: Peskine-Szpiro $\Rightarrow H_Y^{n-j}(X, \mathcal{O}_X) = 0 \ \ \forall\, j < \mathrm{depth}(S/I)$. $\square$

That is false in characteristic 0: consider $R = (k[\mathbb{X}]/I_t(\mathbb{X}))_{(\mathbb{X})}$ where $\mathbb{X}$ is an $m \times n$-matrix of indeterminates. By Bruns-Schwänzl:

$$\lambda_{i,j}(Z) = \begin{cases} 0 & \text{if } j < t^2 - 1 \\ 0 & \text{if } j = t^2 - 1 \text{ and } i > 0 \\ 1 & \text{if } j = t^2 - 1 \text{ and } i = 0 \\ ??? & \text{otherwise} \end{cases}$$

But R is Cohen-Macaulay of dimension $(t-1)(m+n-t+1)$.

Vanishing of $\lambda_{i,j}$ from the depth

Conjecture: $\lambda_{i,j}(Z) = 0 \quad \forall\, j < \mathrm{depth}(R)$ and $i \geq j - 2$.

For example, according to this conjecture, the Lyubeznik table of a 7-dimensional local ring of depth 6 should look like:

$$\Lambda(Z) = \begin{pmatrix} 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

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Vanishing from the depth

Proposition: The above conjecture is equivalent to show that, if $\mathrm{depth}(S/I) \geq 3$, then:

$$H_Y^{n-2}(X, \mathcal{O}_X) = H_Y^{n-1}(X, \mathcal{O}_X) = H_Y^n(X, \mathcal{O}_X) = 0.$$

Proof: $\Rightarrow$: In any case $H_Y^{n-2}(X, \mathcal{O}_X)$ is supported only at the maximal ideal of S, so $H_Y^{n-2}(X, \mathcal{O}_X) \cong E(k)^s$ (Lyubeznik), so $\lambda_{0,2}(Z) = s$. For the converse implication argue like in the proof of few slides above. $\square$

Vanishing from the depth

Theorem (-): If Y is a closed subset of $\mathbb{A}^n$ defined by a graded ideal and such that $\mathrm{depth}(O_Y(Y)) \geq 3$, then

$$H_Y^{n-2}(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}) = H_Y^{n-1}(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}) = H_Y^{n}(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}) = 0.$$

Equivalently, if U is an open subset of $\mathbb{P}^n$ such that the coordinate ring of the complement has depth at least 3, then

$$H^{n-2}(U, \mathcal{F}) = H^{n-1}(U, \mathcal{F}) = H^{n}(U, \mathcal{F}) = 0$$

for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$.

Vanishing from the depth

The coordinate ring of the Segre product

$$\mathbb{P}^1 \times \mathbb{P}^2 \subseteq \mathbb{P}^5$$

is a Cohen-Macaulay 4-dimensional graded ring. However, one can show that, in characteristic 0, there is $m \in \mathbb{Z}$ such that:

$$H^2(\mathbb{P}^5 \setminus (\mathbb{P}^1 \times \mathbb{P}^2), \mathcal{O}_{\mathbb{P}^5}(m)) \neq 0.$$

Set-theoretically Cohen-Macaulayness

Corollary: Let V be a smooth projective variety with nonzero irregularity over a field of characteristic 0. Then there is no projective scheme wich is arithmetically Cohen-Macaulay and set-theoretically the same as V.

To my knowledge, the first example of an irreducible variety not "set-theoretically Cohen-Macaulay" has been exhibited in 2004 by Singh-Walther by using reduction to characteristic p methods.

Question: Are there analog examples for connected curves? Is there a graded ideal $I \subseteq \mathbb{C}[a, b, c, d]$ defining set-theoretically

$$X = \{[s^4, s^3t, st^3, t^4] : [s, t] \in \mathbb{P}^1\} \subseteq \mathbb{P}^3$$

such that $\mathbb{C}[a, b, c, d]/I$ is Cohen-Macaulay?

THANK YOU !!!