Du Bois singularities, positivity and Frobenius

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Notation

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where $h(t) = h_0 + h_1 t + h_2 t^2 + \ldots + h_s t^s \in \mathbb{Z}[t]$ is the *h-polynomial* of R. We will name the coefficients vector $(h_0, h_1, h_2, \ldots, h_s)$ the *h-vector* of R.

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If R is Cohen-Macaulay (CM), then it is easy to see that $h_i \geq 0$ for all $i \geq 0$, however without the CM assumption things get complicated. In the first part of the talk I want to discuss conditions on R and/or on X which ensure at least the nonnegativity of the first h_i's and that the degree of $X \subset \mathbb{P}^n$ is bounded below by their sum.

Serre conditions

For $r \in \mathbb{N}$, we say that R *satisfies the Serre condition* (S_r) if:

$$\operatorname{depth} R_{\mathfrak{p}} \geq \min\{\dim R_{\mathfrak{p}}, r\} \quad \forall\, \mathfrak{p} \in \operatorname{Spec} R.$$

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It turns out that this is equivalent to $\operatorname{depth} R \geq \min\{\dim R, r\}$ and

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In particular, if X is nonsingular, R satisfies the Serre condition (S_r) if and only if $\operatorname{depth} R \geq \min\{\dim R, r\}$.

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In particular, if X is nonsingular, R satisfies the Serre condition (S_r) if and only if $\operatorname{depth} R \geq \min\{\dim R, r\}$. So, the fact that there is some embedding of X such that R satisfies the Serre condition (S_r) is equivalent to $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < r-1$.

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Question

If R satisfies (S_r), is it true that $h_i \geq 0$ for all $i = 0, \ldots, r$?

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The above question is known to have a positive answer for Stanley-Reisner rings R (i.e. if R is defined by squarefree monomial ideals) by a result of Murai and Terai ...

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Such an R can be even chosen to be Buchsbaum of Castelnuovo-Mumford regularity $\operatorname{reg} R = 1$. So the question must be adjusted:

Question

If R satisfies (S_r) and X has nice singularities, is it true that $h_i \geq 0$ for all $i = 0, \ldots, r$?

The main result

Theorem (Dao-Ma-_)

Let R satisfy Serre condition (S_r). Suppose either

- K has characteristic 0 and X is Du Bois, or
- K has positive characteristic and X is globally F-split.

Then $h_i \geq 0$ for all $i = 0, \ldots, r$ and the degree of $X \subset \mathbb{P}^n$ is at least $h_0 + h_1 + \ldots + h_{r-1}$.

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Then $h_i \geq 0$ for all $i = 0, \ldots, r$ and the degree of $X \subset \mathbb{P}^n$ is at least $h_0 + h_1 + \ldots + h_{r-1}$. Furthermore, if $\operatorname{reg} R < r$, or if $h_i = 0$ for some $1 \leq i \leq r$, then R is Cohen-Macaulay.

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- If $\mathrm{char}(K) = 0$, X nonsingular $\Rightarrow X$ Du Bois.
- If $\mathrm{char}(K) > 0$, R SR ring $\Rightarrow X$ globally F-split.
- If $\mathrm{char}(K) > 0$, X nonsingular $\not\Rightarrow X$ globally F-split.

Du Bois singularities by examples

The definition of Du Bois variety (over a field of characteristic 0) is quite involved, it is worth, however, to notice the following:

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- X locally Stanley-Reisner $\Rightarrow$ X Du Bois.
- R Stanley-Reisner ring $\Rightarrow$ X locally Stanley-Reisner.
- X nonsingular $\Rightarrow$ X locally Stanley-Reisner.
- X has simple normal crossing singularities $\Rightarrow$ X locally SR.

Globally F-split varieties

If K has characteristic $p > 0$, X is globally F-split if the Frobenius

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Example (Fedder-Knutson)

Let $K = \mathbb{Z}/p\mathbb{Z}$ and $X \subset \mathbb{P}^n$ be a complete intersection defined by homogeneous polynomials $f_1, \ldots, f_m \in K[X_0, \ldots, X_n]$ such that $\sum_{i=1}^m \deg(f_i) \leq n + 1$.

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(A result of Schwede implies that, in characteristic 0, if infinitely many reductions modulo primes X_p are F-split, then X is Du Bois).

The condition MT_r

Let M be a finitely generated graded S-module, where $S = K[X_0, \ldots, X_n]$. We say that M satisfies the condition MT_r if

$$\operatorname{reg} \operatorname{Ext}_S^{n+1-i}(M, \omega_S) \leq i - r \quad \forall \ i = 0, \ldots, \dim M - 1.$$

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This notion is good for several reasons:

- The condition MT_r does not depend on S.
- The condition MT_r is preserved by taking general hyperplane sections.
- The condition MT_r is preserved by saturating.

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- $h_r + h_{r+1} + \ldots + h_s \geq 0$, or equivalently the multiplicity of M is at least $h_0 + h_1 + \ldots + h_{r-1}$.

Furthermore, if $\operatorname{reg} M < r$ or M is generated in degree 0 and $h_i = 0$ for some $i \leq r$, then M is Cohen-Macaulay.

The condition MT_r

Let $\dim_K R_1 = n+1$ and $S = K[X_0, \ldots, X_n]$. We want to show that, if R satisfies Serre condition (S_r), then it also satisfies MT_r, namely

$$\operatorname{reg} \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \leq i - r \quad \forall\ i = 0, \ldots, \dim R - 1,$$

provided X is Du Bois (in characteristic 0) or globally F-split (in positive characteristic). By the previous lemma this would imply the desired result...

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1. R satisfies Serre condition (S_r).
2. $\dim \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \leq i - r \quad \forall\ i \in \mathbb{N}$.

Regularity of Ext's

So, under our assumptions on X, in order to prove that R satisfies MT_r provided it satisfies (S_r), it is enough to show that

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We show more:

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Let $\mathfrak{m}$ be the homogeneous maximal ideal of S and assume either

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Then $H^j_{\mathfrak{m}}(\operatorname{Ext}^i_S(R, \omega_S))_{>0} = 0$ for all $i, j \in \mathbb{N}$.

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Then $H^j_{\mathfrak{m}}(\operatorname{Ext}^i_S(R,\omega_S))_{>0} = 0$ for all $i,j \in \mathbb{N}$. In particular, $\operatorname{reg} \operatorname{Ext}^i_S(R,\omega_S) \leq \dim \operatorname{Ext}^i_S(R,\omega_S)$ for all $i \in \mathbb{N}$.

A corollary

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Then, the minimal generators of I of degree r are $\leq \binom{c+r-1}{r}$; if equality holds, then R is Cohen-Macaulay.

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Then, the minimal generators of I of degree r are $\leq \binom{c+r-1}{r}$; if equality holds, then R is Cohen-Macaulay.

If $r = 2$, the above corollary is true just assuming that X is reduced...

Frobenius and squarefree initial ideals

Let us fix K of positive characteristic. If X is globally F-split, then the Frobenius F acts injectively on $H^i(X.\mathcal{O}_X)$ for all i.

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One could try to introduce the notion of *globally F-injective*: X is globally F-injective if the Frobenius acts injectively on $H^i(X.\mathcal{O}_X)$ for all i. This is not a very good definition, because it is not equivalent to R being F-injective (i.e. F acts injectively on the local cohomology modules of R), even if the embedding $X \subset \mathbb{P}^n$ is projectively normal.

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Koley, _

If the defining ideal of $X \subset \mathbb{P}^n$ admits a squarefree initial ideal, then X is globally F-injective.

Frobenius and squarefree initial ideals

Conjecture (Constantinescu, De Negri,_)

If X is smooth and the defining ideal of $X \subset \mathbb{P}^n$ admits a squarefree initial ideal, then $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.

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By the previous result with Koley, in characteristic 0, if the defining ideal of $X \subset \mathbb{P}^n$ admits a squarefree initial ideal, then it is not difficult to see that X_p must be globally F-injective for almost all reductions modulo prime numbers p.

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Question

If X is smooth over $\mathbb{Q}$, is it true that X_p is not globally F-injective for infinitely many reductions modulo prime numbers p?

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Conjecture (Constantinescu, De Negri,_)

If X is smooth and the defining ideal of $X \subset \mathbb{P}^n$ admits a squarefree initial ideal, then $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.

By the previous result with Koley, in characteristic 0, if the defining ideal of $X \subset \mathbb{P}^n$ admits a squarefree initial ideal, then it is not difficult to see that X_p must be globally F-injective for almost all reductions modulo prime numbers p.

Question

If X is smooth over $\mathbb{Q}$, is it true that X_p is not globally F-injective for infinitely many reductions modulo prime numbers p? And is it true that X_p is globally F-injective for infinitely many reductions modulo prime numbers p?

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To be honest, in my opinion the question is more interesting than the conjecture. THANK YOU FOR THE ATTENTION.