Cohomological and projective dimensions

COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

Definitions

Let R be a ring, $I \subset R$ an ideal and M an R-module. By

$$H_I^i(M)$$

we mean the ith local cohomology module of M with support in I. One way to think at it is by the following isomorphism:

$$H_I^i(M) \cong \varinjlim \mathrm{Ext}_R^i(R/I_n, M)$$

where $(I_n)_{n\in\mathbb{N}}$ is an inverse system of ideals cofinal with $(I^n)_{n\in\mathbb{N}}$:

$$\forall\ n \in \mathbb{N} \quad I_{n+1} \subset I_n \quad \text{and} \quad \exists\ k, m \in \mathbb{N} : I_k \subset I^n \text{ and } I^m \subset I_n$$

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The cohomological dimension of I is the numerical invariant:

$$\mathrm{cd}(R,I) = \inf\{c \in \mathbb{N} : H_I^i(M) = 0 \ \ \forall\ M \text{ and } i > c\}.$$

It is not difficult to prove that:

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The very starting results are due to Grothendieck:

$$\mathrm{ht}(I) \leq \mathrm{cd}(R,I) \leq \dim R.$$

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Let $R = K[x_1, \ldots, x_n]$ and $I = (f_1, \ldots, f_r) \subset R$ graded.

Peskine-Szpiro: If $\mathrm{char}(K) = p > 0$, then

$$\mathrm{cd}(R, I) \leq \mathrm{pd}(R/I) = n - \mathrm{depth}(R/I)$$

The proof is easy; for all $e \in \mathbb{N}$:

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If $\mathrm{char}(K) = 0$ the above argument of course is not applicable. Actually, if I is the ideal of t-minors of the generic $r \times s$ matrix:

$$\text{Bruns-Schwänzl: } \mathrm{cd}(I) = rs - t^2 + 1$$

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On the other hand $\mathrm{pd}(R/I) = (r - t + 1)(s - t + 1)$, so:

$$\mathrm{cd}(R, I) > \mathrm{pd}(R/I) \text{ (a part from trivial cases).}$$

As one can check, for all $p \leq n - 4$, this provides examples of graded ideals $I \subset R$ for which $\mathrm{cd}(R, I) > \mathrm{pd}(R/I) = p$.

QUESTION: If $\mathrm{pd}(R/I) \leq n - 3$, is $\mathrm{cd}(R, I) \leq n - 3$???

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Their proof relies on the fact that such an R/I has F-pure type.

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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

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PROOF OF THE MAIN RESULT

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Let $I \subset R = K[x_1, \ldots, x_n]$, $X = \operatorname{Proj}(R/I) \subset \mathbb{P}^{n-1}$ and $U = \mathbb{P}^{n-1} \setminus X$.
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Hartshorne: $H^i_{DR}(X) \cong H^i(X_h, \mathbb{C})$ (singular cohomology).

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Thus $H^0_{DR}(X) \cong H^0(X_h, \mathbb{C}) \cong \mathbb{C}$ ((ii) ✓).

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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

Trascendental methods

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$$\exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} f'^m/m! = f' \cdot (1 + \sum_{m \geq 1} f'^m/(m+1)!) \in \mathfrak{a}.$$

The element $g = \sum_{m \geq 1} f'^m/(m+1)! \in \sqrt{\mathfrak{a}}$. This means that $1 + g$ is invertible in A, so f' is actually an element of $\mathfrak{a}$.

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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

The exponential sequence

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The exponential sequence yields a long exact sequence of abelian groups:

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By GAGA $H^0(X_h, \mathcal{O}_{X_h}) \cong H^0(X, \mathcal{O}_X)$, and the latter is an artinian $\mathbb{C}$-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So $H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}^*_{X_h})$ is surjective, and thus

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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

The conclusion

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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

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Using again GAGA, $H^1(X_h, \mathcal{O}_{X_h}) \cong H^1(X, \mathcal{O}_X)$. Moreover

$$H^1(X, \mathcal{O}_X) \cong H^2_{\mathfrak{m}}(R/I)_0$$

where $R = \mathbb{C}[x_1, \ldots, x_n]$, $I \subset R$ is such that $X \cong \mathrm{Proj}(R/I)$ and $\mathfrak{m}$ is the maximal irrelevant. Our assumption was $\mathrm{depth}(R/I) \geq 3$. In particular $H^2_{\mathfrak{m}}(R/I) = 0$, so $H^1(X_h, \mathcal{O}_{X_h}) = 0$. Eventually, by the injection $H^1(X_h, \mathbb{Z}_{X_h}) \hookrightarrow H^1(X_h, \mathcal{O}_{X_k})$ we deduce $H^1(X_h, \mathbb{Z}) \cong H^1(X_h, \mathbb{Z}_{X_h}) = 0$. By the universal coefficient theorem

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and this was the missing piece (iii) to infer $\mathrm{cd}(R, I) \leq n - 3$. $\square$

COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

The conclusion

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