Cohomological and projective dimensions

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\operatorname{depth}(R / I) \geq 2 \Longrightarrow \operatorname{depth}\left(R / I \otimes_{K} \bar{K}\right) \geq 2
$$

So, by a result of Hartshorne, $\operatorname{Proj}\left(R / I \otimes_{K} \bar{K}\right)$ is connected, i.e. $\operatorname{Proj}(R / I)$ is geometrically connected. So we infer $\operatorname{cd}(R, I) \leq n-2$.

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The conclusion

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H^{1}\left(X_{h}, \mathbb{Z}_{X_{h}}\right) \rightarrow H^{1}\left(X_{h}, \mathcal{O}_{X_{h}}\right) \rightarrow H^{1}\left(X_{h}, \mathcal{O}_{X_{h}}^{*}\right) \rightarrow \ldots
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By GAGA $H^{0}\left(X_{h}, \mathcal{O}_{X_{h}}\right) \cong H^{0}\left(X, \mathcal{O}_{X}\right)$, and the latter is an artinian

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By GAGA $H^{0}\left(X_{h}, \mathcal{O}_{X_{h}}\right) \cong H^{0}\left(X, \mathcal{O}_{X}\right)$, and the latter is an artinian $\mathbb{C}$-algebra.
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The exponential sequence yields a long exact sequence of abelian groups:

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## COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

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Using again GAGA, $H^{1}\left(X_{h}, \mathcal{O}_{X_{h}}\right) \cong H^{1}\left(X, \mathcal{O}_{X}\right)$. Moreover

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H^{1}\left(X, \mathcal{O}_{X}\right) \cong H_{m}^{2}(R / I)_{0}
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where $R=\mathbb{C}\left[x_{1}, \ldots, x_{n}\right], I \subset R$ is such that $X \cong \operatorname{Proj}(R / I)$ and $m$
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H^{1}\left(X_{h}, \mathbb{C}\right)=0,
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## COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

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