THE j-MULTIPLICITY OF RATIONAL NORMAL SCROLLS

Matteo Varbaro

Joint with Jack Jeffries and Jonathan Montaño

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The analytic spread of I is by definition:

$$\ell(I) = \dim F(I) \leq \dim G(I) = \dim R = d.$$

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One can easily realize that: $\quad \mathrm{e}(I) = \lim_{s\to\infty} \frac{d!}{s^d} \lambda_R(R/I^{s+1})$.

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$$\varepsilon(I) = \limsup_{s\to\infty} \frac{d!}{s^d} \lambda_R(H^0_{\mathfrak{m}}(R/I^{s+1})).$$

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(iv) (CuHàSrinivasanTheodorescu). Example of irrational ε-multiplicity (defining ideal of a smooth curve in $\mathbb{P}^3$).

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In this talk, we will give a formula for the j-multiplicity of any ideal defining a rational normal scroll.

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If furthermore $[(I^s)^{\mathrm{sat}}]_r = [I^s]_r$ for all $s \gg 0$ and $r \geq st$, then

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It might be helpful to remark that, if R is a polynomial ring and I is minimally generated by degree t polynomials $f_1, \ldots, f_r$, then:

$$F(I) \cong K[f_1, \ldots, f_r].$$

Rational normal scrolls

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We will denote such an ideal by $I(\mathbf{a})$. By definition $I(\mathbf{a})$ has generators in degree 2, so either $\mathrm{j}(I(\mathbf{a})) = 0$, or there is an integer $k \geq 2$ such that:

$$\mathrm{j}(I(\mathbf{a})) = k \cdot \mathrm{e}(F(I(\mathbf{a}))).$$

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Cor (JeMo–): The j-multiplicity of the ideal defining a rational normal scroll of dimension r and codimension c is:

$$\begin{cases} 0 & \text{if } c < r+3\,, \\ 2\left(\binom{2c-4}{c-2} - \binom{2c-4}{c-1}\right) & \text{if } c = r+3\,, \\ 2\left(\sum_{j=2}^{c-r-1} \binom{c+r-1}{c-j} - \binom{c+r-1}{c-1}(c-r-2)\right) & \text{if } c > r+3\,. \end{cases}$$