GRADED BETTI NUMBERS OF COMPONENTWISE LINEAR IDEALS

JÜRGEN HERZOG, LEILA SHARIFAN, AND MATTEO VARBARO

ABSTRACT. In this paper, we study the Betti tables of homogeneous ideals in a polynomial ring. Especially, we concentrate ourselves on componentwise linear ideals, so that, exploiting the deformation to the generic initial ideal, we can reduce the problem to study the Betti tables of strongly stable ideals. We obtain a complete numerical characterization of the graded Betti numbers of ideals with linear resolution, giving two different proofs. We provide a necessary (in general not sufficient) numerical condition for a table being the Betti table of a componentwise linear ideal. Such a condition leads to a characterization of the Betti tables of componentwise linear ideals in three variables. Furthermore we identify the Betti tables of Gotzmann ideals. Eventually, provided the characteristic of the base field is 0, we succeed to characterize the possible extremal Betti numbers (values as well as positions) of any homogeneous ideal.

INTRODUCTION

Minimal free resolutions of modules over a polynomial ring are a classical and fascinating subject. After the work of Boij and Söderberg [BS], and successively of Eisenbud and Schreyer [ES], investigations on Betti tables of graded modules have become one of the central topics of research in Commutative Algebra. In this paper we are going to study the Betti tables of certain classes of ideals of a polynomial ring, and, where it is possible, to give a numerical characterization of them. Even if related, the present article differs substantially in its scopes and methods from the mentioned works. Before explaining the results and the techniques, it is convenient to give a list of the main results of the paper:

(a) A complete characterization of the Betti tables of ideals with linear resolution (Theorem 3.2);
(b) A necessary condition (in general not sufficient, see Example 5.4) for the Betti tables of componentwise linear ideals (Theorem 5.3). This leads to a complete characterization of the Betti tables of componentwise linear ideals in 3 variables (Corollary 5.7);
(c) A complete characterization of the Betti tables of Gotzmann ideals (after Theorem 5.10);
(d) A complete characterization, in characteristic 0, of the extremal Betti numbers (dimensions and positions) of any homogeneous ideal (Theorem 6.7).

Ideals with d-linear resolutions, i.e. generated in degree d and with all the syzygies linear, have been introduced by Eisenbud and Goto in [EG], and since then came up in a lot of situations. As we announced, we will give a complete characterization of the possible Betti tables of an ideal $I \subset P = K[x_1,\ldots,x_n]$ with d-linear resolution in Theorem 3.2. More than tables, in this case we should speak about vectors, since the Betti table of an ideal with d-linear resolution has only one nonzero row, namely:

$$(\beta_{0,d}, \beta_{1,d+1}, \ldots, \beta_{n-1,n-1+d}).$$

To describe the possible Betti tables of ideals with d-linear resolution is equivalent to characterize the possible Betti tables of a strongly stable monomial ideal generated in degree d, essentially thanks to a result of Aramova, Herzog and Hibi [AHH] that allows us to study the generic initial ideal of I. (We have such a reduction also in positive characteristic, but the argument in this case is a bit trickier, see Proposition 3.1). If I is a strongly stable ideal generated in one degree, thanks to the Eliahou-Kervaire formula [EK], to know the Betti numbers of I is equivalent to know the numerical invariants:

$$m_i(I) = |\{u \in G(I) \ : \ x_i|u \text{ and } x_j \nmid u \ \ \forall\, j > i\}|, \quad i = 1,\ldots,n,$$

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where $G(I)$ denotes the set of minimal monomial generators of I. We will show that $(m_1,\dots,m_n)$ corresponds to a strongly stable ideal generated in degree d if and only if it is an O-sequence with $m_2 \leq d$. We give two different proofs of this result.

In Section 2, our approach to analyze the possible m_i's of a strongly stable ideal generated in degree d, uses a quite unconventional multiplication $*$ on the dth graded component S_d of the polynomial ring $S = K[x_0,x_1,x_2,\dots]$ in a countable number of variables. Given two monomials u and v in S_d, we write them as $u = x_{i_1}\cdots x_{i_d}$ with $i_1 \leq \dots \leq i_d$ and $v = x_{j_1}\cdots x_{j_d}$ with $j_1 \leq \dots \leq j_d$, define their product as $u*v = x_{i_1+j_1}x_{i_2+j_2}\cdots x_{i_d+j_d}$ and extend $*$ to all S_d by K-linearity. The K-vector space S_d equipped with this multiplication, denoted $\mathscr{S}_d$, turns out to be isomorphic as a K-algebra with the polynomial ring in d variables, see Proposition 2.1. We characterize the monomial ideals of $\mathscr{S}_d$ in terms of S_d in Lemma 2.5, and it turns out that strongly stable monomial spaces in S_d give rise to ideals of $\mathscr{S}_d$. Furthermore the Hilbert function of $\mathscr{S}_d$ modulo the obtained ideal remembers the m_i's of the starting monomial space, so Macaulay's theorem is the ingredient to conclude. (During the proof we will introduce the notion of *piecewise lexsegment* monomial space, which will reveal itself a crucial concept throughout the paper).

An alternative proof of Theorem 3.2 is given in Section 4, where we define for each strongly stable monomial ideal $I \subset P$ generated in degree $\leq m$ a strongly stable ideal $I^{\text{dual}} \subset K[x_1,\dots,x_m]$ generated in degree $\leq n$ such that $I = (I^{\text{dual}})^{\text{dual}}$. This duality operator, based on Alexander duality and Kalai's stretching operator, establishes a bijection between strongly stable monomial ideals $I \subset K[x_1,\dots,x_n]$ generated in degree $\leq m$ and strongly stable ideals $I^{\text{dual}} \subset K[x_1,\dots,x_m]$ generated in degree $\leq n$, satisfying the additional property that $I \subset P$ has a d-linear resolution if and only if $K[x_1,\dots,x_d]/I^{\text{dual}}$ is Cohen–Macaulay, see Theorem 4.1 and Corollary 4.2. When we wrote the present paper, we were not aware that Murai proved the same characterization of the Betti tables of strongly stable ideals, and so Theorem 3.2, in [Mu]. His proof is similar to the one just described, whereas the approach of Section 2 is different.

In Section 5 we attempt to give a similar explicit characterization of the possible graded Betti numbers of componentwise linear ideals, introduced by Herzog and Hibi in [HH1]. Again, such an issue is equivalent to characterize the graded Betti numbers of strongly stable ideals (not necessarily generated in a single degree). A remark of Murai, 5.1, shows that there are almost no constraints for the *total* Betti numbers of a strongly stable monomial ideal. The situation for the *graded* Betti numbers is much harder to describe. We denote by $I_{\langle j\rangle}$ the ideal generated by the jth graded component I_j of a strongly stable ideal I, set $\mu_{ij}(I) = m_i(I_{\langle j\rangle})$ and define the matrix $\mathscr{M}(I) = (\mu_{ij}(I))$, which we call the *matrix of generators* of I. As explained in the beginning of Section 5, the matrix $\mathscr{M}(I)$ and the graded Betti numbers of I determine each other. Thus we are lead to characterize the integer matrices (μ_{ij}) for which there exists a strongly stable ideal I such that $\mathscr{M}(I) = (\mu_{ij})$. Some necessary conditions for (μ_{ij}) being the matrix of generators for some strongly stable ideal are provided in Theorem 5.3. Unfortunately these conditions are not sufficient to describe the matrices of generators of strongly stable ideals, as shown in Example 5.4. The difficulty of the task of characterizing Betti tables of componentwise linear ideals is also shown by Example 5.5, where we exhibit a noncomponentwise linear ideal with the same Betti table of a componentwise linear ideal, answering negatively a question raised by Nagel and Römer in [NR]. After discussing the main obstruction to construct strongly stable ideals with prescribed matrix of generators, we give sufficient conditions for a matrix to be of the form $\mathscr{M}(I)$ where I is strongly stable in Proposition 5.6. As a consequence it is shown in Corollary 5.7 that the necessary conditions given in Theorem 5.3 are also sufficient when dealing with strongly stable ideals in three variables. Another instance for which the matrix of generators of a particular class of strongly stable ideals can be described, is given in Theorem 5.10, which gives the possible matrices of generators of lexsegment ideals. Then it is explained how to deduce a characterization of the Betti tables of Gotzmann ideals.

Though a complete characterization of the possible Betti numbers of a strongly stable ideal seems to be quite difficult, we succeed in Section 6 to characterize all possible extremal Betti numbers of any homogeneous ideal $I \subset P = K[x_1,\dots,x_n]$, provided that K has characteristic 0. According to Bayer, Charalambous and Popescu [BCP], a Betti number $\beta_{i,i+j} \neq 0$ of I is called extremal if $\beta_{k,k+l} = 0$ for all pairs $(k,l) \neq (i,j)$ with $k \geq i$ and $l \geq j$. It is shown in [BCP] that the positions as well as the values of the extremal Betti numbers of a graded ideal are preserved under taking the generic initial ideal with respect to the reverse lexicographical order. Thus assuming that the base field is of characteristic 0 we may restrict our attention to characterize the extremal Betti numbers of strongly stable ideals. For componentwise linear ideals the restriction on the characteristic is not required. More precisely, let $i_1 < i_2 < \cdots < i_k < n$, $j_1 > j_2 > \cdots > j_k$ and $b_1, \dots, b_k$ be sequences of positive integers. In Theorem 6.7 we give numerical conditions which are equivalent to the property that there exists a componentwise linear (or a strongly stable) ideal I whose extremal Betti numbers are precisely $\beta_{i_p, i_p+j_p}(I) = b_p$ for $p = 1, \dots, k$. In characteristic 0 this gives the possible extremal Betti numbers for any graded ideal.

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1. Terminology

Throughout we denote by $\mathbb{N}$ the set of the natural numbers $\{0, 1, 2, \dots\}$ and by n a positive natural number. We will essentially work with the polynomial rings

$$S = K[x_i : i \in \mathbb{N}]$$

and

$$P = K[x_1, \dots, x_n],$$

where the x_i's are variables over a field K. The reason why we consider a polynomial ring in infinite variables is that it is more natural to deal with it in Section 2, when we will define the $*$-operation. However, for the applications of the theory to the graded Betti numbers, P will be considered. To do not make too heavy the notation, we will introduce the following notions just relatively to S, also if we will use them also for P.

The ring S is graded on $\mathbb{N}$, namely $S = \bigoplus_{d \in \mathbb{N}} S_d$ where

$$S_d = \langle x_{i_1} x_{i_2} \cdots x_{i_d} \ : \ i_1 \le i_2 \le \ldots \le i_d \ \text{ are natural numbers} \rangle.$$

Given a monomial $u \in S_d$, with $d \ge 1$, we set:

$$m(u) = \max\{e \in \mathbb{N} \ : \ x_e \ \text{ divides } \ u\}. \tag{1}$$

A *monomial space* $V \subset S$ is a K-vector subspace of S which has a K-basis consisting in monomials of S. If $V \subset S_d$, we will refer to the *complementary monomial space* V^c *of* V as the K-vector space generated by the monomials of S_d which are not in V. Given a monomial space $V \subset S$ and two natural numbers i, d, such that $d \ge 1$, we set:

$$w_{i,d}(V) = |\{u \text{ monomials in } V \cap S_d \ : \ m(u) = i\}|.$$

Without taking in consideration the degrees,

$$w_i(V) = |\{u \text{ monomials in } V \ : \ m(u) = i\}|.$$

We order the variables of S by the rule

$$x_i > x_j \iff i < j,$$

so that $x_0 > x_1 > x_2 > \dots$. On the monomials, unless we explicitly say differently, we use a degree lexicographical order with respect to the above ordering of the variables. Therefore, given monomials $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ with $i_1 \le i_2 \le \ldots \le i_d$ and $v = x_{j_1} x_{j_2} \cdots x_{j_e}$ with $j_1 \le j_2 \le \ldots \le j_e$, we have:

$$u > v \quad \iff \quad d > e \ \text{ or } \ d = e \ \text{ and } \ \exists\, \ell \in \{1, \dots, d\} : i_k = j_k \ \forall\, k < \ell \text{ and } i_\ell < j_\ell.$$

A monomial space $V \subset S$ is called *stable* if for any monomial $u \in V$, then $(u/x_{m(u)}) \cdot x_i \in V$ for all $i < m(u)$. It is called *strongly stable* if for any monomial $u \in V$ and for each $j \in \mathbb{N}$ such that x_j divides u, then $(u/x_j) \cdot x_i \in V$ for all $i < j$. Obviously a strongly stable monomial space is stable.

The remaining definitions of this section will be given for P, since we do not need them for S. A monomial space $V \subset P$ is called *lexsegment* if, for all $d \in \mathbb{N}$, there exists a monomial $u \in P_d$ such that

$$V \cap P_d = \langle v \in P_d : v \ge u \rangle.$$

We will sometimes denote by:

$$L_{\ge u} = \{v \in P_d : v \ge u\}.$$

Clearly, a lexsegment monomial space is strongly stable. The celebrated theorem of Macaulay explains when a lexsegment monomial space is an ideal. We remind that given a natural number a and a positive integer d, the dth *Macaulay representation* of a is the unique writing:

$$a = \sum_{i=1}^{d} \binom{k(i)}{i} \quad \text{such that } \; k(d) > k(d-1) > \ldots > k(1) \geq 0,$$

see [BH, Lemma 4.2.6]. Then:

$$a^{\langle d \rangle} = \sum_{i=1}^{d} \binom{k(i)+1}{i+1}.$$

A numerical sequence $(h_i)_{i\in\mathbb{N}}$ is called *O-sequence* if $h_0 = 1$ and $h_{d+1} \leq h_d^{\langle d \rangle}$ for all $d \geq 1$. (The reader should be careful because the definition of O-sequence depends on the numbering: A vector $(m_1,\ldots,m_n)$ will be a O-sequence if $m_1 = 1$ and and $m_{i+1} \leq m_i^{\langle i-1 \rangle}$ for all $i \geq 2$). The theorem of Macaulay (for example see [BH, Theorem 4.2.10]) says that, given a numerical sequence $(h_i)_{i\in\mathbb{N}}$, the following are equivalent:

- (i) $(h_i)_{i\in\mathbb{N}}$ is an O-sequence with $h_1 \leq n$.
- (ii) There is a homogeneous ideal $I \subset P$ such that $(h_i)_{i\in\mathbb{N}}$ is the Hilbert function of P/I.
- (iii) The lexsegment monomial space $L \subset P$ such that $L \cap P_d$ consists in the biggest $\binom{n+d-1}{d} - h_d$ monomials, is an ideal.

For any $\mathbb{Z}$-graded finitely generated P-module M, there is a minimal graded free resolution:

$$0 \to \bigoplus_{j\in\mathbb{Z}} P(-j)^{\beta_{p,j}(M)} \to \bigoplus_{j\in\mathbb{Z}} P(-j)^{\beta_{p-1,j}(M)} \to \ldots \to \bigoplus_{j\in\mathbb{Z}} P(-j)^{\beta_{0,j}(M)} \to M \to 0,$$

where $P(k)$ denotes the P-module P supplied with the new grading $P(k)_i = P_{k+i}$. The celebrated Hilbert's Syzygy theorem (for example see [BH, Corollary 2.2.14 (a)]) guarantees $p \leq n$. The natural numbers $\beta_{i,j} = \beta_{i,j}(M)$ are numerical invariants of M, and they are called the *graded Betti numbers* of M. The coarser invariants $\beta_i = \beta_i(M) = \sum_{j\in\mathbb{Z}} \beta_{i,j}$ are called the *(total) Betti numbers* of M. We will refer to the matrix $(\beta_{i,j})$ as the *Betti table* of M. Actually, in the situations we will consider in this paper $M = I$ is a homogeneous ideal of P. In this case $\beta_{i,j} = 0$ whenever $i \geq n$ or $j \leq i$ (unless $I = P$). We will present the Betti table of I as follows:

$$\begin{pmatrix} \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots & \cdots & \beta_{n-1,n} \\ \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \cdots & \cdots & \beta_{n-1,n+1} \\ \beta_{0,3} & \beta_{1,4} & \beta_{2,5} & \cdots & \cdots & \beta_{n-1,n+2} \\ \vdots & \vdots & \vdots & \ldots & \ldots & \vdots \end{pmatrix}.$$

Also if the definition of the Betti table of M predicts infinite many rows, only a finite number of entries are not zero (because M is finitely generated). Consequently, only a finite number of rows are significant, and in the examples we will present throughout the paper we will draw just the significant rows.

For an integer d, the P-module M is said to have a *d-linear resolution* if $\beta_{i,j}(M) = 0$ for every $i = 0,\ldots,p$ and $j \neq i+d$; equivalently, if $\beta_i(M) = \beta_{i,i+d}(M)$ for any $i = 0,\ldots,p$. Notice that if M has d-linear resolution, then it is generated in degree d. The P-module M is said *componentwise linear* if $M_{\langle d \rangle}$ has d-linear resolution for all $d \in \mathbb{Z}$, where $M_{\langle d \rangle}$ means the P-submodule of M generated by the elements of degree d of M. It is not difficult to show that if M has a linear resolution, then it is componentwise linear.

We introduce the following numerical invariants of a $\mathbb{Z}$-graded finitely generated P-module M: For all $i = 1,\ldots,n+1$ and $d \in \mathbb{Z}$:

$$m_{i,d}(M) = \sum_{k=0}^{n} (-1)^{k-i+1} \binom{k}{i-1} \beta_{k,k+d}(M). \tag{2}$$

The following lemma shows that to know the $m_{i,d}(M)$'s is equivalent to know the Betti table of M.

Lemma 1.1. *Let M be a $\mathbb{Z}$-graded finitely generated P-module. Then:*

$$\beta_{i,i+d}(M) = \sum_{k=i}^{n+1} \binom{k-1}{i} m_{k,d}(M). \tag{3}$$

Proof. Set $m_{k,d} = m_{k,d}(M)$ and $\beta_{i,j} = \beta_{i,j}(M)$. By the definition of the $m_{k,d}$'s we have the following identity in $\mathbb{Z}[t]$:

$$\sum_{k=1}^{n+1} m_{k,d} t^{k-1} = \sum_{i=0}^{n} \beta_{i,i+d}(t-1)^i.$$

Replacing t by $s+1$, we get the identity of $\mathbb{Z}[s]$

$$\sum_{k=1}^{n+1} m_{k,d}(s+1)^{k-1} = \sum_{i=0}^{n} \beta_{i,i+d} s^i,$$

that implies the lemma. ☐

Let us define also the coarser invariants:

$$m_i(M) = \sum_{d\in\mathbb{Z}} m_{i,d}(M) \quad \forall\, i = 1,\dots,n+1. \tag{4}$$

If $M = I$ is a homogeneous ideal of P, notice that $m_{i,d} = 0$ if $i = n+1$ or $d < 0$. We say that a monomial ideal $I \subset P$ is stable (strongly stable) (lexsegment) if the underlining monomial space is. By $G(I)$, we will denote the unique minimal set of monomial generators of I. If I is a stable monomial ideal, we have the following nice interpretation by the Eliahou-Kervaire formula [EK] (see also [HH2, Corollary 7.2.3]):

$$\begin{aligned} m_{i,d}(I) = w_{i,d}(\langle G(I)\rangle) &= |\{u \text{ monomials in } G(I)\cap P_d \;:\; m(u) = i\}| \\ m_i(I) = w_i(\langle G(I)\rangle) &= |\{u \text{ monomials in } G(I) \;:\; m(u) = i\}|. \end{aligned} \tag{5}$$

From Lemma 1.1 and (5) follows that a stable ideal generated in degree d has a d-linear resolution. Furthermore, if I is a stable ideal, then $I_{\langle d\rangle}$ is stable for all natural numbers d. So any stable ideal is componentwise linear.

When $M = I$ is a stable monomial ideal we will consider (5) the definition of the $m_{i,d}$'s, and we will refer to (3) as the Eliahou-Kervaire formula.

2. The $*$-operation on monomials and strongly stable ideals

We are going to give a structure of associative commutative K-algebra to the K-vector space S_d, in the following way: Given two monomials u and v in S_d, we write them as $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1 \le i_2 \le \ldots \le i_d$ and $v = x_{j_1}x_{j_2}\cdots x_{j_d}$ with $j_1 \le j_2 \le \ldots \le j_d$. Then we define their product as

$$u * v = x_{i_1+j_1}x_{i_2+j_2}\cdots x_{i_d+j_d}.$$

We can extend $*$ to the whole S_d by K-linearity. It is straightforward to check that $*$ is associative and commutative. Therefore K embeds in S_d by means of the rule $\lambda \mapsto \lambda x_0^d$. We will denote by $\mathscr{S}_d$ the K-vector space S_d supplied with such an algebra structure. Actually $\mathscr{S}_d$ has a natural graded structure: In fact, we can write $\mathscr{S}_d = \oplus_{e\in\mathbb{N}}(\mathscr{S}_d)_e$ where

$$(\mathscr{S}_d)_e = \langle u \text{ monomial of } S_d \;:\; m(u) = e\rangle.$$

Notice that $(\mathscr{S}_d)_0 = \langle x_0^d\rangle \cong K$ and that $(\mathscr{S}_d)_e$ is a finite dimensional K-vector space. Therefore, $\mathscr{S}_d$ is actually a positively graded K-algebra. Moreover, if $u = x_0^{a_0}\cdots x_e^{a_e} \in \mathscr{S}_d$, with $a_e \neq 0$ and $e \geq 1$. Then

$$u = (x_0^{a_0}x_1^{a_1+\ldots a_e}) * (x_0^{a_0+a_1}x_1^{a_2+\ldots a_e}) * \ldots * (x_0^{a_0+\ldots+a_{e-1}}x_1^{a_e}),$$

so $\mathscr{S}_d$ is a standard graded K-algebra, that is $\mathscr{S}_d = K[(\mathscr{S}_d)_1]$. Particularly, $\mathscr{S}_d$ is Noetherian. Notice that $(\mathscr{S}_d)_1$ is a K-vector space of dimension d, namely:

$$(\mathscr{S}_d)_1 = \langle x_0^{d-1}x_1, x_0^{d-2}x_1^2, \ldots, x_1^d\rangle.$$

Actually, we are going to prove that $\mathscr{S}_d$ is a polynomial ring in d variables over K.

Proposition 2.1. *The ring $\mathscr{S}_d$ is isomorphic, as a graded K-algebra, to the polynomial ring in d variables over K.*

Proof. Let $K[y_1,\ldots,y_d]$ be the polynomial ring over K in d variables. Of course there is a graded surjective homomorphism of K-algebras ϕ from $K[y_1,\ldots,y_d]$ to $\mathscr{S}_d$, by extending the rule:

$$\phi(y_i) = x_0^{i-1}x_1^{d+1-i}. \tag{6}$$

In order to show that ϕ is an isomorphism, it suffices to exhibit an isomorphism of K-vector spaces between the graded components of $\mathscr{S}_d$ and $K[y_1,\ldots,y_d]$. To this aim pick a monomial $u \in (\mathscr{S}_d)_e$:

$$u = x_0^{a_0}\cdots x_e^{a_e}, \quad a_i \in \mathbb{N},\ a_e > 0 \ \text{ and } \ \sum_{i=0}^{e} a_i = d.$$

To such a monomial we associate the monomial of $K[y_1,\ldots,y_d]_e$

$$y_{a_0+1}y_{a_0+a_1+1}\cdots y_{a_0+\ldots+a_{e-1}+1}.$$

It is easy to see that the above application is one-to-one, so the proposition follows. $\square$

Remark 2.2. For the sequel it is useful to familiarize with the map ϕ. For instance, one can easily verify that:

$$\phi(y_1^{b_1}y_2^{b_2}\cdots y_d^{b_d}) = x_{b_1}x_{b_1+b_2}\cdots x_{b_1+\ldots+b_d}. \tag{7}$$

Proposition 2.1 guarantees that ϕ has an inverse, that we will denote by $\psi = \phi^{-1} : \mathscr{S}_d \to K[y_1,\ldots,y_d]$. As one can show:

$$\psi(x_0^{a_0}x_1^{a_1}\cdots x_e^{a_e}) = y_{a_0+1}y_{a_0+a_1+1}\cdots y_{a_0+\ldots+a_{e-1}+1}. \tag{8}$$

Given a monomial space V, of course we have an isomorphism of K-vector spaces

$$V \cong \mathscr{S}_d/V^c.$$

However in general the above isomorphism does not yield a structure of K-algebra to V, because V^c may be not an ideal of $\mathscr{S}_d$. We are interested to characterize those monomials spaces $V \subset S_d$ such that V^c is an ideal of $\mathscr{S}_d$. For what follows it is convenient to introduce the following definition.

Definition 2.3. Let $V \subset S$ be a monomial space. We will call it *block stable* if for any $u = x_0^{a_0}\cdots x_e^{a_e} \in V$ and for any $i = 1,\ldots,e$, we have that

$$\frac{u}{x_i^{a_i}\cdots x_e^{a_e}} \cdot x_{i-1}^{a_i}\cdots x_{e-1}^{a_e} \in V.$$

Remark 2.4. Notice that a strongly stable monomial space is also stable and block stable. On the other side block stable monomial spaces might be not stable (it is enough to consider $\langle x_0^2,\, x_1^2\rangle$). There are also stable monomial spaces which are not block stable: Consider the monomial space:

$$V = \langle x_0^3,\, x_0^2x_1,\, x_0x_1^2,\, x_0x_1x_2,\, x_0x_1x_3\rangle \subset S_3.$$

It turns out that V is stable, but not block stable, because

$$\frac{x_0x_1x_3}{x_1x_3} \cdot x_0x_2 = x_0^2x_2 \notin V.$$

Eventually, the monomial space $\langle x_0^3,\, x_0^2x_1,\, x_0x_1^2,\, x_0x_1x_2\rangle \subset S_3$ is both stable and block stable, but is not strongly stable.

Lemma 2.5. *Let $V \subset S_d$ be a monomial space. Then V is block stable if and only if V^c is an ideal of $\mathscr{S}_d$.*

Proof. "Only if"-part. Consider a monomial $u \in V^c$. By contradiction there is $i \in \{1,\ldots,d-1\}$ such that

$$w = u * (x_0^i x_1^{d-i}) \notin V^c.$$

If $u = x_{p_1}\cdots x_{p_d}$ with $p_1 \le \ldots \le p_d$, then

$$w = x_{p_1}\cdots x_{p_i}\cdot x_{p_{i+1}+1}\cdots x_{p_d+1}.$$

Since V is block stable and w is a monomial of V, then

$$u = \frac{w}{x_{p_{i+1}+1}\cdots x_{p_d+1}} \cdot x_{p_{i+1}}\cdots x_{p_d} \in V,$$

a contradiction.

"If"-part. Pick $u = x_0^{a_0} \cdots x_e^{a_e} \in V$. By contradiction there is $i \in \{1, \ldots, e\}$ such that

$$w = \frac{u}{x_i^{a_i} \cdots x_e^{a_e}} \cdot x_{i-1}^{a_i} \cdots x_{e-1}^{a_e} \notin V.$$

Since V^c is an ideal of $\mathscr{S}_d$ and $w \in V^c$, we have

$$u = w * (x_0^{a_1+\ldots+a_{i-1}} x_1^{a_i+\ldots+a_e}) \in V^c.$$

This contradicts the fact that we took $u \in V$. $\square$

The following corollary, essentially, is why we introduced $\mathscr{S}_d$.

Corollary 2.6. *Let $(w_i)_{i\in\mathbb{N}}$ be a sequence of natural numbers. If there exists a strongly stable monomial space $V \subset S_d$ (actually it is enough that V is block stable) such that $w_i(V) = w_i$ for any $i \in \mathbb{N}$, then $(w_i)_{i\in\mathbb{N}}$ is an O-sequence such that $w_1 \le d$.*

Proof. That $w_0 = 1$ and $w_1 \le d$ is clear. By Lemma 2.5 V^c is an ideal of $\mathscr{S}_d$. So, Proposition 2.1 implies that $\mathscr{S}_d/V^c$ is a standard graded K-algebra. Clearly we have

$$\mathrm{HF}_{\mathscr{S}_d/V^c}(i) = w_i(V) = w_i \quad \forall\, i \in \mathbb{N},$$

(HF denotes the Hilbert function) so we get the conclusion by the theorem of Macaulay. $\square$

The above corollary can be reversed. To this aim we need to understand the meaning of "strongly stable" in $\mathscr{S}_d$. By Proposition 2.1 $\mathscr{S}_d \cong K[y_1, \ldots, y_d]$, so we already have a notion of "strongly stable" in $\mathscr{S}_d$. However, we want to describe it in terms of the multiplication $*$.

Lemma 2.7. *Let W be a monomial space of $K[y_1, \ldots, y_d]$. We recall the isomorphism $\phi : K[y_1, \ldots, y_d] \to \mathscr{S}_d$ of (6). The following are equivalent:*

(i) *W is a strongly stable monomial space.*
(ii) *If $x_0^{a_0} \cdots x_e^{a_e} \in \phi(W)$ with $a_e > 0$, then $x_0^{a_0} \cdots x_i^{a_i-1} \cdot x_{i+1}^{a_{i+1}+1} \cdots x_e^{a_e} \in \phi(W)$ for all $i \in \{0, \ldots, e-1\}$ such that $a_i > 0$.*

Proof. (i) $\implies$ (ii). If $u = x_0^{a_0} x_1^{a_1} \cdots x_e^{a_e} \in \phi(W)$ with $a_e > 0$, then

$$\psi(u) = y_{a_0+1} y_{a_0+a_1+1} \cdots y_{a_0+\ldots+a_{e-1}+1} \in W,$$

see (8). Since W is strongly stable, then for all $i \in \{0, \ldots, e-1\}$:

$$w = y_{a_0+1} \cdots y_{a_0+\ldots+(a_i-1)+1} \cdot y_{a_0+\ldots+(a_i-1)+(a_{i+1}+1)+1} \cdots y_{a_0+\ldots+a_{e-1}+1} \in W.$$

Therefore, if $a_i > 0$, we get $v = x_0^{a_0} \cdots x_i^{a_i-1} \cdot x_{i+1}^{a_{i+1}+1} \cdots x_e^{a_e} = \phi(w)$, so $v \in \phi(W)$.

(ii) $\implies$ (i). Let $w = y_1^{b_1} y_2^{b_2} \cdots y_d^{b_d} \in W$. Then, using (7),

$$\phi(w) = x_{b_1} x_{b_1+b_2} \cdots x_{b_1+\ldots+b_d} \in \phi(W).$$

By contradiction there exist p and q in $\{1, \ldots, d\}$ such that $b_p > 0$, $q < p$ and

$$\frac{w}{y_p} \cdot y_q = y_1^{b_1} \cdots y_q^{b_q+1} \cdots y_p^{b_p-1} \cdots y_d^{b_d} \notin W.$$

Of course we can suppose that $q = p-1$, so we get a contradiction, because the assumptions yield:

$$\phi\left(\frac{w}{y_p} \cdot y_{p-1}\right) = x_{b_1} \cdots x_{b_1+\ldots+(b_{p-1}+1)} x_{b_1+\ldots+(b_{p-1}+1)+(b_p-1)} \cdots x_{b_1+\ldots+b_d} \in \phi(W).$$

$\square$

Thanks to Lemma 2.7, therefore, it will be clear what we mean for a monomial space of $\mathscr{S}_d$ being strongly stable.

Proposition 2.8. *Let $V \subset S_d$ be a monomial space. The following are equivalent:*

(i) *V^c is a strongly stable monomial subspace of $\mathscr{S}_d$;*
(ii) *V is a strongly stable monomial subspace of S_d.*

Proof. First we prove (i) $\implies$ (ii). Pick $u = x_0^{a_0} \cdots x_e^{a_e} \in V$. By contradiction, assume that there exists $i \in \{1, \ldots, e\}$ such that $w = x_0^{a_0} \cdots x_{i-1}^{a_{i-1}+1} x_i^{a_i-1} \cdots x_e^{a_e} \notin V$. So $w \in V^c$, and since V^c is a strongly stable monomial ideal of $\mathscr{S}_d$, by Lemma 2.7 we get $u \in V^c$, which is a contradiction.

(ii) $\implies$ (i). By Lemma 2.5 we have that V^c is an ideal of $\mathscr{S}_d$. Consider $u = x_0^{a_0} \cdots x_e^{a_e} \in V^c$ with $a_e > 0$ and $i \in \{0, \ldots, e-1\}$. If $w = x_0^{a_0} \cdots x_i^{a_i-1} \cdot x_{i+1}^{a_{i+1}+1} \cdots x_e^{a_e}$ were not in V^c, then u would be in V because V is a strongly stable monomial space. Thus V^c has to be strongly stable once again using Lemma 2.7. $\square$

Theorem 2.9. *Let $(w_i)_{i\in\mathbb{N}}$ be a sequence of natural numbers. Then the following are equivalent:*

(i) *There exists a strongly stable monomial space $V \subset S_d$ such that $w_i(V) = w_i$ for any $i \in \mathbb{N}$.*
(ii) *There exists a block stable monomial space $V \subset S_d$ such that $w_i(V) = w_i$ for any $i \in \mathbb{N}$.*
(iii) *$(w_i)_{i\in\mathbb{N}}$ is an O-sequence such that $w_1 \leq d$.*

Proof. (i) $\implies$ (ii) is obvious and (ii) $\implies$ (iii) is Corollary 2.6. So (iii) $\implies$ (i) is the only thing we still have to prove. If the sequence $(w_i)_{i\in\mathbb{N}}$ satisfies the conditions of (iii), then the theorem of Macaulay guarantees that there exists a lexsegment ideal $J \subset K[y_1, \ldots, y_d]$ such that

$$\mathrm{HF}_{K[y_1,\ldots,y_d]/J}(i) = w_i \quad \forall\, i \in \mathbb{N}$$

Being a lexsegment ideal, J is strongly stable. So $\phi(J)^c$ is a strongly stable monomial subspace of S_d by Proposition 2.8. Clearly we have:

$$m_i(\phi(J)^c) = \mathrm{HF}_{K[y_1,\ldots,y_d]/J}(i) = w_i \quad \forall\, i \in \mathbb{N},$$

thus we conclude. $\square$

Actually, a careful reading of the proof of Theorem 2.9 shows that, given a O-sequence, we can give explicitly a strongly stable monomial subspace $V \subset S_d$ such that $w_i(V) = w_i$ for any $i \in \mathbb{N}$. The reason is that to any Hilbert function is associated a unique lexsegment ideal: Let $(w_i)_{i\in\mathbb{N}}$ be a sequence of natural numbers. For any $i \in \mathbb{N}$, set

$$V_i = \{\text{biggest } w_i \text{ monomials } u \in S_d \text{ such that } m(u) = i\}.$$

Then we call $V = \langle \cup_{i\in\mathbb{N}} V_i \rangle \subset S_d$ the *piecewise lexsegment* monomial space (of type $(d, (w_i)_{\mathbb{N}})$). The proof of Theorem 2.9 yields:

Corollary 2.10. *The piecewise lexsegment of type $(d, (w_i)_{\mathbb{N}})$ is strongly stable if and only if $(w_i)_{\mathbb{N}}$ is a O-sequence such that $w_1 \leq d$.*

Notice that the established interaction between S_d and $K[y_1, \ldots, y_d]$ can be also formulated between

$$K[x_0, \ldots, x_m] \quad \text{and} \quad K[y_1, \ldots, y_d]/(y_1, \ldots, y_d)^{m+1} \quad \forall\, m \geq 1.$$

Therefore, an interesting corollary of Proposition 2.8 is the following.

Corollary 2.11. *Let us define the sets*

$$A = \{\textit{strongly stable monomial ideals of } K[x_0, \ldots, x_m] \textit{ generated in degree } d\}$$

and

$$B = \{\textit{strongly stable monomial ideals of } K[y_1, \ldots, y_d] \textit{ with height } d \textit{ and generated in degrees} \leq m+1\}.$$

Then the assignation $V \mapsto \psi(V^c)$ establishes a correspondence between A and B.

Proof. Notice that if $I \subset K[y_1, \ldots, y_d]$ is of height d, then $(y_1, \ldots, y_d)^k \subset I$ for all $k \geq \mathrm{reg}(I)$. Since I is generated in degrees $\leq m+1$ and componentwise linear, we have $\mathrm{reg}(I) \leq m+1$, so we are done by what said before the corollary. $\square$

3. The possible Betti numbers of an ideal with linear resolution

We would like to characterize the possible graded Betti numbers of a componentwise linear ideal of $P=K[x_1,\dots,x_n]$. This is a difficult task, in fact we are not going to solve the problem in its full generality. In this section, exploiting the techniques developed in Section 2, we will give a complete characterization when the ideal has a linear resolution. Such an issue is equivalent to characterize the possible graded Betti numbers of a strongly stable monomial ideal of P generated in one degree. Actually, more generally, to characterize the possible Betti tables of a componentwise linear ideal of P is equivalent to characterize the possible Betti tables of a strongly stable monomial ideal of P. In fact, in characteristic 0 this is true because the generic initial ideal of any ideal I is strongly stable [Ei, Theorem 15.23]. Moreover, if I is componentwise linear and the term order is degree reverse lexicographic, then the graded Betti numbers of I are the same of those of $\mathrm{Gin}(I)$ by a result of Aramova, Herzog and Hibi in [AHH]. In positive characteristic it is still true that for a degree reverse lexicographic order the graded Betti numbers of I are the same of those of $\mathrm{Gin}(I)$, provided that I is componentwise linear. But in this case $\mathrm{Gin}(I)$ might be not strongly stable. However, it is known that, at least for componentwise linear ideals, it is stable [CHH, Lemma 1.4]. The graded Betti numbers of a stable ideal do not depend from the characteristic, because the Elihaou-Kervaire formula (3). So to compute the graded Betti numbers of $\mathrm{Gin}(I)$ we can consider it in characteristic 0. Let us call J the ideal $\mathrm{Gin}(I)$ viewed in characteristic 0. The ideal J, being stable, is componentwise linear, so we are done by what said above. Summarizing, we showed:

Proposition 3.1. *The following sets coincide:*

(1) $\{$*Betti tables* $(\beta_{i,j}(I))$ *where* $I\subset P$ *is componentwise linear*$\}$*;*
(2) $\{$*Betti tables* $(\beta_{i,j}(I))$ *where* $I\subset P$ *is strongly stable*$\}$*;*

So, we get the following:

Theorem 3.2. *Let $m_1,\dots,m_n$ be a sequence of natural numbers. Then the following are equivalent:*

(1) *There exists a homogeneous ideal $I\subset P$ with d-linear resolution such that $m_k(I)=m_k$ for all $k=1,\dots,n$;*
(2) *There exists a strongly stable monomial ideal $I\subset P$ generated in degree d such that $m_k(I)=m_k$ for all $k=1,\dots,n$ if and only if:*
(3) *$(m_1,\dots,m_n)$ is an O-sequence such that $m_2\le d$, that is:*
    (a) $m_1=1$*;*
    (b) $m_2\le d$*;*
    (c) $m_{i+1}\le m_i^{\langle i-1\rangle}$ *for any $i=2,\dots,n-1$.*

Proof. By virtue of Proposition 3.1, (1) $\iff$ (2). Moreover, if I is strongly stable, then $m_i(I)=w_i(\langle G(I)\rangle)$ for all $i=1,\dots,n$, see (5). Since the monomial space $\langle G(I)\rangle$ is strongly stable, Theorem 2.9 yields the equivalence (2) $\iff$ (3). $\square$

Example 3.3. Let us see an example: Theorem 3.2 assures that we will never find a homogeneous ideal $I\subset R=K[x_1,x_2,x_3,x_4]$ with minimal free resolution:

$$0\longrightarrow R(-6)^6\longrightarrow R(-5)^{22}\longrightarrow R(-4)^{29}\longrightarrow R(-3)^{14}\longrightarrow I\longrightarrow 0.$$

In fact I, using (2), should satisfy $m_1(I)=1$, $m_2(I)=3$, $m_3(I)=4$ and $m_4(I)=6$. This is not an O-sequence, thus the existence of I would contradict Theorem 3.2.

4. A duality for strongly stable ideals

In this section we define a duality operator which assigns to each strongly stable monomial ideal $I\subset P=K[x_1,\dots,x_n]$ generated in degree $\le m$ a strongly stable ideal $I^{\mathrm{dual}}\subset K[x_1,\dots,x_m]$ such that $I=(I^{\mathrm{dual}})^{\mathrm{dual}}$. This duality will be used to give an alternative proof of Theorem 3.2.

The duality operator is a composition of several operators which we are now going to describe: let $u=x_{i_1}x_{i_2}\cdots x_{i_d}$ be a monomial with $i_1\le i_2\le\ldots\le i_d$. Following Kalai, we define the *stretched* monomial arising from u to be

$$u^\sigma=x_{i_1}x_{i_2+1}\cdots x_{i_d+(d-1)}.$$

Notice that u^σ is a squarefree monomial.

The compress operator τ is inverse to σ. If $v = x_{j_1}x_{j_2}\cdots x_{j_d}$ is a squarefree monomial (therefore $j_1 < j_2 < \ldots < j_d$) we define the *compressed* monomial arising from v to be

$$v^\tau = x_{j_1}x_{j_2-1}\cdots x_{j_d-(d-1)}.$$

Let $I \subset P$ be a strongly stable ideal with $G(I) = \{u_1,\ldots,u_r\}$ and $m = \max_i\{\deg u_i\}$. We set

$$I^\sigma = (u_1^\sigma, u_2^\sigma, \ldots, u_r^\sigma) \subset K[x_1,\ldots,x_{n+m-1}].$$

As shown in [HH2, Lemma 11.2.5], one has that I^σ is a squarefree strongly stable ideal. Recall that a squarefree monomial ideal $J \subset K[x_1,\ldots,x_t]$ is called *squarefree strongly stable*, if for all squarefree generators u of I and all $i < j$ for which x_j divides u and x_i does not divides u, one has that $(u/x_j)\cdot x_i \in J$.

We need one more ingredient to define the dual of a strongly stable ideal: let $J \subset K[x_1,\ldots,x_t]$ be a squarefree monomial ideal. Then J has an unique irredundant presentation $J = P_{F_1} \cap P_{F_2} \cap \ldots \cap P_{F_r}$ where $F_i \subset [t] = \{1,\ldots,t\}$ for all i, and where $P_F = (\{x_j \ \ j \in F\})$ for $F \subset [t]$. We set $J^\vee = (x_{F_1},\ldots,x_{F_r})$ where $x_F = \prod_{j\in F} x_j$ for $F \subset [t]$, and call $J^\vee$ the *Alexander dual* of J. This naming is justified by the fact, that for the Stanley–Reisner ideal I_Δ of a simplicial complex we have that $(I_\Delta)^\vee = I_{\Delta^\vee}$ where $\Delta^\vee$ is the Alexander dual of the simplicial complex Δ, see [HH2, Subsection 1.5.3].

Now we are ready to define the dual of a strongly stable I. We set

$$I^{\text{dual}} = ((I^\sigma)^\vee)^\tau,$$

where for a squarefree monomial ideal J with $G(J) = (u_1,\ldots,u_m)$ we set $J^\tau = (u_1^\tau,\ldots,u_m^\tau)$.

Theorem 4.1. *Let $I \subset P$ be a strongly stable ideal generated in degree $\leq m$. Then*

(a) *$I^{\text{dual}} \subset K[x_1,\ldots,x_m]$, and is generated in degree $\leq n$;*
(b) *I^{dual} is strongly stable;*
(c) *$(I^{\text{dual}})^{\text{dual}} = I$.*

Thus the assignment $I \mapsto I^{\text{dual}}$ establishes a bijection between strongly stable ideals in $K[x_1,\ldots,x_n]$ generated in degree $\leq m$, and strongly stable ideals in $K[x_1,\ldots,x_m]$ generated in degree $\leq n$.

Proof. (a) For a squarefree monomial u it holds that $u^\tau \in K[x_1,\ldots,x_m]$ if $m(u) - \deg u + 1 \leq m$. Thus in order to prove that $I^{\text{dual}} \subset K[x_1,\ldots,x_m]$ we have to show that $m(u) - \deg u + 1 \leq m$ for all $u \in G((I^\sigma)^\vee)$. Since the Alexander dual of a squarefree strongly stable ideal is again squarefree strongly stable, it follows from [HS, Proposition 4.1] that

$$\operatorname{proj\,dim} K[x_1,\ldots,x_{n+m-1}]/(I^\sigma)^\vee = \max\{m(u) - \deg u + 1 : \ u \in G((I^\sigma)^\vee)\}. \tag{9}$$

On the other hand, using a result of Terai obtained in [Te] and the fact that $\beta_{ij}(I) = \beta_{ij}(I^\sigma)$ for all i and j, as shown in [HH1, Lemma 11.2.6], we obtain

$$\begin{aligned} \operatorname{proj\,dim} K[x_1,\ldots,x_{n+m-1}]/(I^\sigma)^\vee &= \operatorname{proj\,dim}(I^\sigma)^\vee + 1 = \operatorname{reg} K[x_1,\ldots,x_{n+m-1}]/I^\sigma + 1 \\ &= \operatorname{reg}(I^\sigma) = \operatorname{reg}(I) \leq m. \end{aligned} \tag{10}$$

For the last inequality we used that for a strongly stable monomial ideal I the highest degree of a generator of I coincides with $\operatorname{reg}(I)$, as follows from the Eliahou-Kervaire formula (3). Combining (9) and (10) we see that $I^{\text{dual}} \subset K[x_1,\ldots,x_m]$.

Similarly one has

$$\begin{aligned} n &\geq \operatorname{proj\,dim} K[x_1,\ldots,x_n]/I = \operatorname{proj\,dim} K[x_1,\ldots,x_{n+m-1}]/I^\sigma \\ &= \operatorname{reg} K[x_1,\ldots,x_{n+m-1}]/(I^\sigma)^\vee + 1 = \operatorname{reg}(I^\sigma)^\vee = \operatorname{reg}(I^{\text{dual}}). \end{aligned} \tag{11}$$

The statements (b) and (c) are obvious by the definition of I^{dual} and by the property of Alexander duality and the stretching and compressing operators. □

The Hilbert series of a graded P-module M is of the form $\operatorname{HS}_M(t) = Q(t)/(1-t)^d$ where $Q(t) = h_0 + h_1 t + \cdots + h_s t^s$ is an integer polynomial with $Q(t) \neq 0$ and where $d = \dim M$, see [BH, Corollary 4.1.8]. The coefficient vector of $Q(t)$ is called the h-vector of M. Observing that the Hilbert series is additive on short exact sequences one obtains that

$$\operatorname{HS}_M(t) = \frac{\sum_{i,j}(-1)^i \beta_{i,j}(M) t^j}{(1-t)^n}. \tag{12}$$

This is implies that the graded Betti numbers of M determine the h-vector.

The following corollary is an immediate consequence of Theorem 4.1 and a famous result of Eagon and Reiner [ER, Theorem 3].

Corollary 4.2. *Let $I \subset P$ be a strongly stable ideal. The following conditions are equivalent:*

(a) *I has a d-linear resolution;*
(b) *I is generated in degree d;*
(c) *$K[x_1,\dots,x_d]/I^{\text{dual}}$ is Cohen–Macaulay.*

If the equivalent conditions hold, then

$$\sum_{i\geq 0}\beta_i(I)t^i=\sum_{j\geq 0}h_j(1+t)^j,$$

where $(h_0,h_1,\dots)$ is the h-vector of $K[x_1,\dots,x_d]/I^{\text{dual}}$.

Proof. The equivalence of (a) and (b) is a consequence of the Eliahou-Kervaire formula (3).

(a) $\Leftrightarrow$ (c): By the result [ER, Theorem 3] of Eagon and Reiner the equivalence of (a) and (c) holds for any squarefree monomial ideal if we replace the dual operator by Alexander duality. Since the stretching operator and the compressing operator, which are inverse to each other, preserve the graded Betti numbers ([HH2, Lemma 11.2.6]) for strongly stable and squarefree strongly stable ideals, respectively, they also preserve the property of having a linear resolution and of being Cohen–Macaulay, and also preserve the h-vectors, as can be easily deduced from (12). Thus all assertions follow from the Eagon–Reiner theorem. $\square$

Let I be a strongly stable ideal generated in one degree d. From the Elihaou-Kervaire formula (3) yields:

$$\sum_{i\geq 0}\beta_i(I)t^i=\sum_{j\geq 0}m_{j+1}(I)(1+t)^j, \tag{13}$$

Comparing (13) with the identity given in Corollary 4.2 we see that $m_{j+1}(I)=h_j$ for all j. Since $(h_0,h_1,\dots)$ is the h-vector of a Cohen–Macaulay ring, the $m_j(I)$ is an O-sequence by Macaulay's theorem. This yields a new proof of Theorem 3.2 (2) $\Rightarrow$ (3).

The converse implication (3) $\Rightarrow$ (2) can also be proved by using this duality: let $m_1,\dots,m_n$ be an O-sequence such that $m_2\leq d$. Then the proof of Macaulay's theorem as given in [BH, Theorem 4.2.10] shows that there exists a lexsegment ideal $I\subset K[x_1,\dots,x_d]$ such that $\dim_K(K[x_1,\dots,x_d]/I)_j=m_{j+1}$ for $j=0,\dots,n-1$, and $\dim_K(K[x_1,\dots,x_d]/I)_j=0$ for $j\geq n$. Since $K[x_1,\dots,x_d]/I$ is of Krull dimension 0, it is in particular Cohen–Macaulay and its h-vector coincides with $(m_1,\dots,m_n)$. Thus Corollary 4.2 implies I^{dual} is a strongly stable ideal generated in degree d with $m_i(I^{\text{dual}})=m_i$ for all i.

Remark 4.3. A similar argument was also used by Murai in [Mu, Proposition 3.8] to achieve Theorem 3.2.

Example 4.4. This example demonstrates the above construction: the sequence of numbers $m_1=1, m_2=3$ and $m_3=5$ satisfy the conditions of Theorem 3.2 (c). The (unique) lexsegment ideal $I\subset K[x_1,x_2,x_3]$ with h-vector $(1,3,5)$ is $I=(x_1^2,x_1x_2^2,x_1x_2x_3,x_1x_3^2,x_2^3,x_2^2x_3,x_2x_3^2,x_3^3)$. Then we get

$$I^\sigma=(x_1x_2,x_1x_3x_4,x_1x_3x_5,x_1x_4x_5,x_2x_3x_4,x_2x_3x_5,x_2x_4x_5,x_3x_4x_5),$$

and

$$(I^\sigma)^\vee=(x_1x_2x_3,x_1x_2x_4,x_1x_2x_5,x_1x_3x_4,x_1x_3x_5,x_1x_4x_5,x_2x_3x_4,x_2x_3x_5,x_2x_4x_5).$$

Finally, we obtain

$$I^{\text{dual}}=((I^\sigma)^\vee)^\tau=(x_1^3,x_1^2x_2,x_1^2x_3,x_1x_2^2,x_1x_2x_3,x_1x_3^3,x_2^3,x_2^2x_3,x_2x_3^2).$$

Remark 4.5. Corollary 4.2 implies that the assignment $I\mapsto I^{\text{dual}}$ establishes the bijection given in Corollary 2.11 between strongly stable ideals in $P=K[x_1,\dots,x_n]$ generated in degree d, and strongly stable ideals in $K[x_1,\dots,x_d]$ of height d generated in degree $\leq n$. This happens because if $I\subset P$ is a strongly stable ideal generated in degree d then by [HH2, Lemma 11.2.6] and equation (10),

$$\operatorname{proj}\dim(K[x_1,\dots,x_d]/I^{\text{dual}})=\operatorname{proj}\dim(K[x_1,\dots,x_{d+n-1}]/(I^\sigma)^\vee)=\operatorname{reg}(I)=d.$$

So $\dim(K[x_1,\cdots,x_d]/I^{\text{dual}}) = \operatorname{depth}(K[x_1,\cdots,x_d]/I^{\text{dual}}) = 0$ and the conclusion follows.

It is worth remarking that this bijection actually coincides with the one described in Corollary 2.11. In the sense that, if $J \subset K[x_0,\cdots,x_{n-1}]$ is a strongly stable ideal generated in degree d and $J' \subset K[x_1,\ldots,x_n]$ is the ideal J under the transformation $x_i \mapsto x_{i+1}$, then

$$\psi(\langle G(J)^c\rangle) = J'^{\text{dual}},$$

where $\psi(\langle G(J)^c\rangle)$ is considered in the x_i's, and not in the y_i's. To show this equality, it is enough to prove that $J'^{\text{dual}} \subset \psi(\langle G(J)^c\rangle)$ because the graded rings $K[x_1,\cdots,x_d]/J'^{\text{dual}}$ and $K[x_1,\cdots,x_d]/\langle G(J)^c\rangle$ share the same Hilbert function.

By contrary assume that there exists $u \in G(J'^{\text{dual}}) \setminus \langle G(J)^c\rangle$. So there is $v = x_{\ell_1}\cdots x_{\ell_e} \in G((J'^{\sigma})^{\vee})$ and $w = x_0^{\alpha_1}\cdots x_f^{\alpha_{f+1}} \in G(J)$ such that $u = v^{\tau} = \psi(w)$. So

$$x_{\ell_1}x_{\ell_2-1}\cdots x_{\ell_e-(e-1)} = x_{\alpha_1+1}x_{\alpha_1+\alpha_2+1}\cdots x_{\alpha_1+\cdots+\alpha_f+1}.$$

This means that $e = f$ and $v = x_{\alpha_1+1}x_{\alpha_1+\alpha_2+2}\cdots x_{\alpha_1+\cdots+\alpha_e+e}$. So

$$J'^{\sigma} \subset (x_{\alpha_1+1},x_{\alpha_1+\alpha_2+2},\cdots,x_{\alpha_1+\cdots+\alpha_e+e}).$$

By definition of J' we have $w' = x_1^{\alpha_1}\cdots x_{e+1}^{\alpha_{e+1}} \in G(J')$, but

$$w'^{\sigma} = \prod_{i=1}^{\alpha_1} x_i \cdot \prod_{i=\alpha_1+1}^{\alpha_1+\alpha_2} x_{i+1}\cdots \prod_{i=\alpha_1+\ldots+\alpha_e+1}^{\alpha_1+\ldots+\alpha_{e+1}} x_{i+e} \notin (x_{\alpha_1+1},x_{\alpha_1+\alpha_2+2},\cdots,x_{\alpha_1+\cdots+\alpha_e+e}),$$

which is a contradiction.

5. Graded Betti numbers of componentwise linear ideals

In this section we want to discuss the problem of characterizing the graded Betti numbers of a componentwise linear ideal $I \subset P = K[x_1,\ldots,x_n]$. By Proposition 3.1, to this purpose we can assume that I is strongly stable. So, by Eliahou-Kervaire (3), we have to consider the possible matrices $(m_{i,j}(I))$ where I ranges over the strongly stable ideals. Actually, we will formulate the results with respect to another matrix, that will be denoted by $(\mu_{i,j}(I))$. As we will see soon, to know $(m_{i,j}(I))$ or $(\mu_{i,j}(I))$ is equivalent.

Before beginning the discussion on graded Betti numbers, we want to show that to characterize the total Betti numbers of a componentwise linear ideal is an easy task. Using Proposition 3.1 and (3), it is enough to characterize the possible sequences $(m_1(I),\ldots,m_n(I))$ where I is a strongly stable ideal. The following remark, due to Satoshi Murai, yields the answer:

Remark 5.1. (Murai). Let $(m_1,\ldots,m_n)$ be a sequence of natural numbers. The following are equivalent:

(i) $m_1 = 1$ and $m_{i+1} = 0$ whenever $m_i = 0$.

(ii) There exists a strongly stable ideal $I \subset P$ such that $m_i(I) = m_i$ for any $i = 1,\ldots,n$.

That (ii) $\Longrightarrow$ (i) is very easy to show. For the reverse implication, given a sequence $(m_1,\ldots,m_n)$ satisfying (i), set $k = \max_{\ell}\{m_\ell \neq 0\}$. By assumption we have $m_i \geq 1$ for all $i = 1,\ldots,k$, therefore it makes sense to define the following monomials spaces for each $j = 1,\ldots,k-1$:

$$V_j = \left\langle \prod_{i=1}^{j-2} x_i^{m_{i+1}-1}\cdot x_{j-1}^{m_j-1}x_j^{m_{j+1}},\ \ \prod_{i=1}^{j-2} x_i^{m_{i+1}-1}\cdot x_{j-1}^{m_j-2}x_j^{m_{j+1}+1},\ \ \ldots,\ \ \prod_{i=1}^{j-2} x_i^{m_{i+1}-1}\cdot x_j^{m_j+m_{j+1}-1} \right\rangle.$$

We also define:

$$V_k = \left\langle \prod_{i=1}^{k-2} x_i^{m_{i+1}-1}\cdot x_{k-1}^{m_k-1}x_k,\ \ \prod_{i=1}^{k-2} x_i^{m_{i+1}-1}\cdot x_{k-1}^{m_k-2}x_k^2,\ \ \ldots,\ \ \prod_{i=1}^{k_2} x_i^{m_{i+1}-1}\cdot x_k^{m_k} \right\rangle.$$

Clearly, for all $j = 1,\ldots,k$, we have $w_i(V_j) = m_j$ if $i = j$ and $w_i(V_j) = 0$ otherwise. Set:

$$I = \left(\bigoplus_{j=1}^{k} V_k\right) \subset P.$$

It is easy to see that I is a strongly stable monomial ideal and that $\langle G(I)\rangle = \bigoplus_{j=1}^{k} V_k$, so we conclude.

Let $I \subset P$ be a strongly stable monomial ideal. Notice that both $I_{\langle j\rangle}$ and $\mathfrak{m}I$, where j is a natural number and $\mathfrak{m} = (x_1,\dots,x_n)$ is the graded maximal ideal of P, are strongly stable. For all $j \in \mathbb{N}$ and $i = 1,\dots,n$, we define:

$$\mu_{i,j}(I) = m_i(I_{\langle j\rangle}).$$

As we said in the beginning of this section, to know the matrix $(m_{i,j}(I))$ or $(\mu_{i,j}(I))$ are equivalent issues. To see it, notice that if $J \subset P$ is a strongly stable monomial ideal, then for all $i = 1,\dots,n$:

$$m_i(\mathfrak{m}J) = \sum_{q=1}^{i} m_q(J).$$

Therefore we have the formula:

$$m_{i,j}(I) = m_i(I_{\langle j\rangle}) - m_i(\mathfrak{m}I_{\langle j-1\rangle}) = \mu_{i,j}(I) - \sum_{q=1}^{i} \mu_{q,j-1}(I) \tag{14}$$

that implies that we can pass from the $\mu_{i,j}$'s to the $m_{i,j}$'s. From it follows also that we can do the converse path by induction on j, because $\mu_{i,d}(I) = m_{i,d}(I)$ if d is the smallest degree in which I is not zero. Therefore, using Proposition 3.1, to characterize the possible Betti tables of the componentwise linear ideals is equivalent to answer the following question:

Question 5.2. *What are the possible matrices $\mathscr{M}(I) = (\mu_{i,j}(I))$ where $I \subset P$ is a strongly stable ideal?*

We will refer to $\mathscr{M} = \mathscr{M}(I)$ as the *matrix of generators* of the strongly stable ideal I. We will feature $\mathscr{M}$ as follows:

$$\mathscr{M} = \begin{pmatrix} \mu_{1,1} & \mu_{2,1} & \mu_{3,1} & \cdots & \cdots & \mu_{n,1} \\ \mu_{1,2} & \mu_{2,2} & \mu_{3,2} & \cdots & \cdots & \mu_{n,2} \\ \mu_{1,3} & \mu_{2,3} & \mu_{3,3} & \cdots & \cdots & \mu_{n,3} \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \end{pmatrix}$$

We can immediately state the following:

Theorem 5.3. *Let $\mathscr{M} = (\mu_{i,j})$ be the matrix of generators of a strongly stable monomial ideal $I \subset P$. Then the following conditions hold:*

(i) *Each non-zero row vector $(\mu_{1,j},\mu_{2,j},\dots,\mu_{n,j})$ of $\mathscr{M}$ is an O-sequence such that $\mu_{2,j} \le j$.*
(ii) *For all i and j one has $\mu_{i,j} \ge \sum_{q=1}^{i} \mu_{q,j-1}$.*

Proof. Condition (i) follows from Theorem 3.2 since $I_{\langle j\rangle}$ has a j-linear resolution for all j greater than or equal to the lower degree in which I is not zero. Condition (ii) follows from (14). □

Notice that the Noetherianity of P (or if you prefer conditions (i) and (ii) of Theorem 5.3) implies that there exists $m \in \mathbb{N}$ such that $\mu_{i,j}(I) = \sum_{q=1}^{i} \mu_{q,j-1}(I)$ for all $j > m$ and $i \in \{1,\dots,n\}$. So, though $\mathscr{M}$ has infinitely many rows, the relevant ones are just a finite number, and in the examples we will write just them.

One may expect that the conditions described in Theorem 5.3 are sufficient. But this is not the case at all:

Example 5.4. One obstruction is illustrated already by Remark 5.1: Consider the matrix

$$\mathscr{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 1 & d & 0 & 0 \\ 1 & d+1 & d+1 & k \end{pmatrix}$$

where the first nonzero row from the top is the dth and $d+1 < k \le (d+1)^{\langle 2\rangle}$. Such a matrix clearly satisfies the necessary conditions of Theorem 5.3. However, if there existed a strongly stable ideal

$I \subset K[x_1,x_2,x_3,x_4]$ with matrix of generators $\mathscr{M}$, then it would satisfy $m_1(I)=1$, $m_2(I)=d$, $m_3(I)=0$ and $m_4(I)=k-d-1>0$, a contradiction to Remark 5.1. The first matrix of this kind is:

$$\mathscr{M}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 4 \end{pmatrix}.$$

The explained obstruction gives rise to a class of counterexamples. However, such a class does not fill the gap between the existence of a strongly stable ideal with matrix of generators $\mathscr{M}$ and the necessary conditions of Theorem 5.3. Let us look at the following matrix.

$$\mathscr{M}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 2 \\ 1 & 4 & 6 & 9 \end{pmatrix}.$$

One can check that the necessary conditions described in Theorem 5.3 hold. However one can show that there is no strongly stable monomial ideal $I \subset K[x_1,\dots,x_4]$ with $\mathscr{M}$ as matrix of generators. Notice that such an ideal would have $m_1(I)=1$, $m_2(I)=3$, $m_3(I)=2$ and $m_4(I)=3$, which does not contradict Remark 5.1.

Example 5.5. Obviously the property of having linear resolution can be detected looking at the graded Betti numbers. In the following example we show that this is not anymore true for componentwise linear ideals, and this strengthens the impression that to give a complete characterization of the possible graded Betti numbers of a componentwise linear ideal is probably a hard task. More precisely, we are going to exhibit two ideals I and J, one componentwise linear and one not, with the same Betti tables. This answers negatively a question raised by Nagel and Römer [NR, Question 1.1].

Consider the ideals of $K[x_1,x_2,x_3]$:

$$I=(x_1^4,\ x_1^3x_2,\ x_1^2x_2^2,\ x_1x_2^3,\ x_2^4,\ x_1^3x_3,\ x_1^2x_2x_3^2,\ x_1^2x_3^3,\ x_1x_2^2x_3^2)$$

and

$$J=(x_1^4,\ x_1^3x_2,\ x_1^2x_2^2,\ x_1^3x_3,\ x_1x_2^2x_3,\ x_1x_2x_3^2,\ x_1x_2^4,\ x_1^2x_3^3,\ x_2^4x_3).$$

Notice that I and J are generated in degrees 4 and 5. By CoCoA [Co] one can check that I and J have the same Betti table, namely:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6 & 6 & 1 \\ 3 & 6 & 3 \end{pmatrix}.$$

One can easily check that I is strongly stable, so in particular it is componentwise linear. On the contrary J is not componentwise linear, since $J_{\langle 4\rangle}=(x_1^4,x_1^3x_2,x_1^2x_2^2,x_1^3x_3,x_1x_2^2x_3,x_1x_2x_3^2)$, as one can check by CoCoA, has not a 4-linear resolution.

We are going to explain some reasons why the conditions of Theorem 5.3 for a matrix $\mathscr{M}$ are in general not sufficient to have a strongly stable ideal corresponding to it. By the discussion after Theorem 2.9, we know that for a given sequences $(m_1,\dots,m_n)$ of integers, there exists a strongly stable monomial ideal J generated in degree d such that $m_i(J)=m_i$, if and only if the piecewise lexsegment ideal of type $(d,(m_1,\dots,m_n))$ is strongly stable. Unfortunately, even if J is a piecewise lexsegment, the ideal $\mathfrak{m}J$ is not necessarily a piecewise lexsegment. For instance, keeping in mind the last matrix in Example 5.4, the piecewise lexsegment ideal of type $(5,(1,3,2,2))$ is:

$$J=(x_1^5,x_1^4x_2,x_1^3x_2^2,x_1^2x_2^3,x_1^4x_3,x_1^3x_2x_3,x_1^4x_4,x_1^3x_2x_4).$$

However $u=x_1^3x_3^3 \notin \mathfrak{m}J$, whereas $v=x_1^2x_2^3x_3 \in \mathfrak{m}J$. Since v is lexicographically smaller than u and $m(u)=m(v)=3$, $\mathfrak{m}J$ is not a piecewise lexsegment ideal. This fact does not make any troubles when the number of variables is at most three, as we can see later. To see this, we need the following more general proposition.

Proposition 5.6. *Let $\mathscr{M} = (\mu_{i,j})$ be a matrix. Then $\mathscr{M}$ is the matrix of generators of a strongly stable monomial ideal $I \subset P$, provided that the following conditions hold:*

(1) *Each nonzero row vector $(\mu_{1,j},\dots,\mu_{n,j})$ of $\mathscr{M}$ is an O-sequence with $\mu_{2,j} \leq j$.*
(2) *For all $i \in \{1,\dots,n\}$ one has $\mu_{i,j} \geq \sum_{q=1}^{i} \mu_{q,j-1}$.*
(3) *If $m \in \mathbb{N}$ is the least number such that $\mu_{i,h} = \sum_{q=1}^{i} \mu_{q,h-1}$ for each $h > m$, then for each $j \leq m$ there exists a strongly stable ideal ${}^{j}I \subset P$ generated in degree j such that $m_i({}^{j}I) = \mu_{i,j}$ for all $i = 1,\dots,n$ and ${}^{j}I \cap K[x_1,\dots,x_{n-1}]_{j+1}$ is a piecewise lexsegment monomial space.*

Proof. Let d be the least natural number such that the row vector $(\mu_{1,d},\dots,\mu_{n,d})$ is nonzero. We claim to have built a strongly stable ideal $I(j) \subset P$ such that $\mu_{i,k}(I(j)) = \mu_{i,k}$ for any $k \leq j$ and $I(j)_{\langle j\rangle} \cap K[x_1,\dots,x_{n-1}] = {}^{j}I \cap K[x_1,\dots,x_{n-1}]$. If $j = m$, then the desired ideal is $I = I(m)$. If not, however we can assume $j \geq d$ ($I(d) = {}^{d}I$). We set

$$m_{i,j+1} = \mu_{i,j+1} - \sum_{q=1}^{i} \mu_{q,j}.$$

Let $L(j+1)$ be the ideal generated by the biggest $m_{i,j+1}$ ($i = 1,\dots,n$) monomials $u \in P_{j+1} \setminus I(j)$ such that $m(u) = i$ (they exist thanks to condition (1)). Set $I(j+1) = I(j) + L(j+1)$. Clearly the first $j+1$ rows of the matrix of generators of $I(j+1)$ coincide with the ones of $\mathscr{M}$. Then notice that $I(j+1)_{\langle j+1\rangle} \cap K[x_1,\dots,x_{n-1}]$ is the piecewise lexsegment of type $(j+1,(\mu_{1,j+1},\cdots,\mu_{n-1,j+1}))$ by construction. Because ${}^{j+1}I \cap K[x_1,\dots,x_{n-1}]_{j+2}$ is a piecewise lexsegment monomial space, ${}^{j+1}I \cap K[x_1,\dots,x_{n-1}]$ is forced to be the piecewise lexsegment of type $(j+1,(\mu_{1,j+1},\cdots,\mu_{n-1,j+1}))$. So we get the equality:

$$I(j+1)_{\langle j+1\rangle} \cap K[x_1,\dots,x_{n-1}] = {}^{j+1}I \cap K[x_1,\dots,x_{n-1}].$$

To conclude the proof, we have to show that $I(j+1)$ is strongly stable. This reduces to show that, if $u \in I(j+1)$ is a monomial of degree $j+1$ with $m(u) = n$, then $(u/x_i)x_k$ belongs to $I(j+1)$ for all $1 \leq k < i \leq n$ such that $x_i \mid u$. We consider two cases. If $u \in I(j)$ we are done, because $I(j) \subset I(j+1)$ is strongly stable. If $u \in L(j+1)$, then we consider the monomial ideal

$$T(j+1) = (v \in G(I(j+1)_{\langle j+1\rangle}) : \ m(v) < n \text{ or } v \geq u) \subset I(j+1).$$

Observe that $T(j+1)$ is a piecewise lexsegment ideal of type $(j+1,(\mu_{1,j+1},\dots,\mu_{n-1,j+1},a))$, where $a \leq \mu_{n,j+1}$. Since $(\mu_{1,j+1},\dots,\mu_{n-1,j+1},a)$ is an O-sequence with $\mu_{2,j+1} \leq j+1$, it follows that $T(j+1)$ is strongly stable by the discussion after Theorem 2.9. Thus for each $1 \leq k < i \leq n$ such that $x_i | u$, we have that $(u/x_i)x_k \in T(j+1) \subset I(j+1)$. □

Corollary 5.7. *Let $\mathscr{M} = (\mu_{i,j})$ be a matrix with 3 columns. Then $\mathscr{M}$ is the matrix of generators of a strongly stable monomial ideal $I \subset K[x_1,x_2,x_3]$ if and only if the following conditions hold:*

(1) *Each non-zero column vector $(\mu_{1,j},\mu_{2,j},\mu_{3,j})$ of $\mathscr{M}$ is an O-sequence with $\mu_{2,j} \leq j$.*
(2) *For all $j \in \mathbb{N}$ one has $\mu_{2,j} \geq \mu_{1,j-1} + \mu_{2,j-1}$ and $\mu_{3,j} \geq \mu_{1,j-1} + \mu_{2,j-1} + \mu_{3,j-1}$.*

Proof. The conditions are necessary from Theorem 5.3. Furthermore, since an ideal $I \subset K[x_1,x_2]$ generated in one degree is piecewise lexsegment if and only if it is strongly stable, we automatically have condition (3) of Proposition 5.6. □

Although the complete characterization of the matrix of generators of an arbitrary strongly stable ideal seems to be very complicated, based on the fact that the lexsegment property of an ideal is preserved under multiplication by the maximal ideal $\mathfrak{m}$, one may expect a characterization for the matrix of generators of lexsegment ideals. For answering this question, first we define the concept of a d-lex sequence.

Definition 5.8. A sequence of non-negative integers $m_1,\cdots,m_n$ is called a d-lex sequence, if there exists a lexsegment ideal $L \subset P$ generated in degree d such that $m_i(L) = m_i$ for all i.

Because if $I \subset P$ is a lexsegment ideal, then $\mathfrak{m}I$ is still a lexsegment ideal, we clearly have that $\mathscr{M} = (\mu_{i,j})$ is the matrix of generators of a lexsegment ideal if and only if the following conditions hold:

(1) Each non-zero column vector $(\mu_{1,j},\mu_{2,j},\dots,\mu_{n,j})$ of $\mathscr{M}$ is a j-lex sequence.
(2) For all i and j one has $\mu_{i,j}\geq \sum_{q=1}^{i}\mu_{q,j-1}$.

Therefore to characterize the matrix of generators of lexsegment ideals we need to characterize arbitrary d-lex sequences. To do this, we have to recall the definition of the natural decomposition of the complement set of monomials belonging to a lexsegment ideal generated in a fixed degree. In what follows we denote by $[x_t,\dots,x_n]_r$ $(1\leq t\leq n)$ the set of all monomials of degree r in the variables $x_t,\dots,x_n$.

Definition 5.9. Let $u=x_{j(1)}\dots x_{j(d)}\in P_d$ $(1\leq j(1)\leq\cdots\leq j(d)\leq n)$ be a monomial and set $L_{<u}=\{v\in P_d \mid v<u\}$. Following the method described in [BH, page 159] (where $L_{<u}$ is denoted by $\mathscr{L}_u$) we can partition the set $L_{<u}$ as:

$$L_{<u}=\bigcup_{i=1}^{d}[x_{j(i)+1},\dots,x_n]_{d-i+1}\cdot x_{j(1)}\cdots x_{j(i-1)},$$

which is called the *natural decomposition* of $L_{<u}$.

Before proving the next result, notice that the powers of the maximal ideal are lexsegment ideals, and the following formula holds for their d-lex sequences:

$$m_i(\mathfrak{m}^d)=\binom{i+d-2}{d-1}. \tag{15}$$

Theorem 5.10. *Let $m_1,\dots,m_n$ be a sequence of natural numbers and let $\mu=\sum_{i=1}^{n}m_i$. Suppose that*

$$\ell=\binom{n+d-1}{d}-\mu=\sum_{i=1}^{d}\binom{k(i)}{i}$$

is the d-th Macaulay representation of ℓ. Then $m_1,\cdots,m_n$ is a d-lex sequence, if and only if

$$m_i=\binom{i+d-2}{d-1}-\sum_{j=1}^{d}\binom{k(j)-n+i-1}{j-1}.$$

Proof. The sequence $m_1,\dots,m_n$ is a d-lex sequence if and only if $I_u=(L_{\geq u})$ satisfies $m_i(I_u)=m_i$ for all $i=1,\dots,n$, where u is the μth biggest monomial of degree d. Let us write $u=x_{j(1)}\cdots x_{j(d)}, 1\leq j(1)\leq\cdots\leq j(d)\leq n$. By the natural decomposition of $L_{<u}$ we have:

$$\ell=|L_{<u}|=\sum_{i=1}^{d}\dim_K[x_{j(i)+1},\dots,x_n]_{d-i+1}=\sum_{i=1}^{d}\binom{n-j(i)+d-i}{d-i+1}.$$

Setting $t=d-i+1$ and $k(t)=n-j(d-t+1)+t-1$, we have that $\sum_{t=1}^{d}\binom{k(t)}{t}$ is the dth Macaulay representation of ℓ. The natural decomposition of $L_{<u}$ and (15) show that

$$m_i((L_{<u}))=\sum_{t=1}^{d}m_i(x_{j(d-t+1)+1},\dots,x_n)^t=\sum_{t=1}^{d}\binom{i-j(d-t+1)+t-2}{t-1}=\sum_{t=1}^{d}\binom{k(t)-n+i-1}{t-1}.$$

Because

$$m_i(I_u)=m_i(\mathfrak{m}^d)-m_i((L_{<u})),$$

we get the conclusion thanks to (15). $\square$

We recall that a homogeneous ideal $I\subset P$ is said to be *Gotzmann* if the number of minimal generators of $\mathfrak{m}I_{\langle j\rangle}$ is the smallest possible for every $j\in\mathbb{N}$, namely equal to:

$$\binom{n+j}{j+1}-\left(\binom{n+j-1}{j}-\mu_j\right)^{\langle j\rangle},$$

where μ_j is the number of minimal generators of $I_{\langle j\rangle}$. The graded Betti numbers of a Gotzmann ideal coincide with its associated lexsegment ideal, see [HH1]. Therefore Theorem 5.10 characterizes also the graded Betti numbers of Gotzmann ideals.

6. The possible extremal Betti numbers of a graded ideal

For a fixed $\ell \in \{1,\dots,n\}$, $d \in \mathbb{N}$ and $k \leq \binom{\ell+d-2}{\ell-1}$, we denote by $u(\ell,k,d)$ the kth biggest monomial $u \in S_d$ such that $m(u) = \ell$. Or, equivalently, x_ℓ times the kth biggest monomial in $K[x_1,\dots,x_\ell]_{d-1}$. By $U(\ell,k,d)$ we denote the ideal of S generated by the set $L_{\geq u(\ell,k,d)} \cap K[x_1,\dots,x_\ell]$. Notice that $U(\ell,k,d)$ is not a lexsegment in S. However, it is the extension of a lexsegment in $K[x_1,\dots,x_\ell]$. Furthermore, $U(\ell,k,d)$ is obviously a piecewise lexsegment in S. In this section we need to introduce the following definition: A monomial ideal $I \subset S$ generated in one degree is called *piecewise lexsegment up to ℓ* if $I \cap K[x_1,\dots,x_\ell] \subset K[x_1,\dots,x_\ell]$ is piecewise lexsegment.

Remark 6.1. Notice that, for all $q \in \mathbb{N}$, denoting by $\mathfrak{m} \subset S$ the maximal irrelevant ideal, $\mathfrak{m}^q U(\ell,k,d) \cap K[x_1,\dots,x_\ell]$ is equal to $U(\ell, m_\ell(\mathfrak{m}^q U(\ell,k,d)), d+q)) \cap K[x_1,\dots,x_\ell]$. In particular, $\mathfrak{m}^q U(\ell,k,d)$ is a piecewise lexsegment up to ℓ.

Lemma 6.2. *The ideal $U(\ell,k,d) \subset S$ is the smallest strongly stable ideal containing the biggest k monomials $u_i \in S_d$ such that $m(u_i) = \ell$ for all $i = 1,\dots,k$.*

Proof. Let $J \subset S$ be the smallest strongly stable ideal containing the biggest k monomials $u_i \in S_d$ such that $m(u_i) = \ell$ for all $i = 1,\dots,k$. Being the extension of a lexsegment, $U(\ell,k,d)$ is strongly stable, so that $J \subset U(\ell,k,d)$. Therefore, let us show the inclusion $U(\ell,k,d) \subset J$. Let u be a minimal monomial generator of $U(\ell,k,d)$. So u has degree d and $m(u) \leq \ell$. Actually, we can assume $m(u) < \ell$, otherwise there is nothing to prove. So let us write:

$$u = x_1^{a_1} \cdots x_{\ell-1}^{a_{\ell-1}}.$$

By definition $u > u(\ell,k,d) = x_1^{b_1} \cdots x_\ell^{b_\ell}$. Set $F = \{i : a_i > b_i\}$. Because $u > u(\ell,k,d)$, we have $F \neq \emptyset$ and $a_j = b_j$ for all $j < i_0 = \min\{i : i \in F\}$. If $|F| = 1$, then $a_i = b_i$ for all $i_0 < i < \ell$ and $b_\ell = a_{i_0} - b_{i_0}$, so that $u = x_{i_0}^{a_{i_0}-b_{i_0}} \cdot (u(\ell,k,d)/x_\ell^{a_{i_0}-b_{i_0}}) \in J$. If $|F| > 1$, take $j > i_0$ such that $a_j > b_j$. The monomial $u' = x_\ell \cdot (u/x_j)$ is such that $u' > u(\ell,k,d)$ and $m(u') = \ell$. Therefore $u' \in J$, so that $u = x_j \cdot (u'/x_\ell)$ belongs to J too. □

The above lemma allows us to characterize the possible extremal Betti numbers of a homogeneous ideal in a polynomial ring. To this aim, we start with a discussion. To $U(\ell,k,d)$ we can associate the numerical sequence $(m_1,\dots,m_\ell)$ where $m_i = m_i(U(\ell,k,d))$. Notice that $m_\ell = k$. By the theory developed in Section 2, if V is a strongly stable monomial ideal generated in degree d such that $m_\ell(V) = k$, then there must exist a strongly stable piecewise lexsegment ideal U such that $m_i(U) = m_i(V)$ and containing the k biggest monomials $u \in S_d$ such that $m(u) = \ell$. By Lemma 6.2 $U(\ell,k,d) \subset U$, so that $m_i \leq m_i(V)$ for all i. It is possible to characterize the possible numerical sequences like these. To this purpose, we need to introduce a notion. Given a natural number a and a positive integer d, consider the dth Macaulay representation of a, say $a = \sum_{i=1}^{d} \binom{k(i)}{i}$. For all integer numbers j, we set:

$$a^{\langle d,j \rangle} = \sum_{i=1}^{d} \binom{k(i)+j}{i+j},$$

where we put $\binom{p}{q} = 0$ whenever p or q are negative, and $\binom{0}{0} = 1$. Notice that $a^{\langle d,0 \rangle} = a$ and $a^{\langle d,1 \rangle} = a^{\langle d \rangle}$.

Lemma 6.3. *If $k \leq \binom{\ell+d-2}{\ell-1}$, then:*

$$m_i(U(\ell,k,d)) = k^{\langle \ell-1, i-\ell \rangle} \quad \forall\, i = 1,\dots,\ell.$$

Furthermore, if $i \geq 2$, then $k^{\langle \ell-1,i-\ell \rangle} = \min\{a : k \leq a^{\langle i-1,\ell-i \rangle}\}$.

Proof. First we will show that, if $i \geq 2$, then:

$$k^{\langle \ell-1,i-\ell \rangle} = \min\{a : k \leq a^{\langle i-1,\ell-i \rangle}\}.$$

Let us consider the $(\ell-1)$th Macaulay representation of k, namely $k = \sum_{j=1}^{\ell-1} \binom{k(j)}{j}$. So

$$b = k^{\langle \ell-1,i-\ell \rangle} = \sum_{j=\ell-i}^{\ell-1} \binom{k(j)+i-\ell}{j+i-\ell}.$$

If $\max\{j : k(j) < j\} \geq \ell - i$, then the above one is the $(i-1)$th Macaulay representation of b: Therefore $b^{\langle i-1,\ell-i\rangle} = k$, so the statement is obvious in this case.

So we can assume that $\max\{j : k(j) < j\} < \ell - i$. In particular, $k(\ell - i) \geq \ell - i$, so that the $(i-1)$th Macaulay representation of $b-1$ is

$$b - 1 = \sum_{j=\ell-i+1}^{\ell-1} \binom{k(j)+i-\ell}{j+i-\ell}.$$

Thus $(b-1)^{\langle i-1,\ell-i\rangle} = \sum_{j=\ell-i+1}^{\ell-1} \binom{k(j)}{j}$, which in this case is smaller than k. So $b \leq \min\{a : k \leq a^{\langle i-1,\ell-i\rangle}\}$. On the other hand, let us consider the $(i-1)$th Macaulay representation of b, namely $b = \sum_{j=1}^{i-1} \binom{h(j)}{j}$. By [BH, Lemma 4.2.7], we infer the inequality

$$(h(i-1),\dots,h(1)) > (k(\ell-1)+i-\ell,\dots,k(\ell-i+1)+i-\ell)$$

in the lexicographical order. Of course the inequality keeps to be true when shifting of $\ell - i$, namely

$$(h(i-1)+\ell-i,\dots,h(1)+\ell-i) > (k(\ell-1),\dots,k(\ell-i+1))$$

in the lexicographical order. Again using [BH, Lemma 4.2.7], we deduce that $b^{\langle i-1,\ell-i\rangle} > k$. So $b \geq \min\{a : k \leq a^{\langle i-1,\ell-i\rangle}\}$, that lets us conclude this part.

Let us prove that

$$m_i(U(\ell,k,d)) = k^{\langle \ell-1,i-\ell\rangle} \ \ \forall\, i = 1,\dots,\ell.$$

The condition $k \leq \binom{\ell+d-1}{\ell}$ assures that we can construct $V = U(\ell,k,d)$. The equality is true for $i=1$, because $k^{\langle \ell-1,1-\ell\rangle} = 1$. From Theorem 3.2 we have, for all $i = 2,\dots,\ell$:

$$m_{i+1}(V) \leq m_i(V)^{\langle i-1\rangle}\ ,\ \ m_{i+2}(V) \leq m_{i+1}(V)^{\langle i\rangle}\ ,\ \ \dots\ ,\ \ k = m_\ell(V) \leq m_{\ell-1}(V)^{\langle \ell-2\rangle}.$$

Putting together the above inequalities, we get:

$$k \leq m_i(V)^{\langle i-1,\ell-i\rangle}.$$

From this and what proved above we deduce that:

$$m_i(V) \geq k^{\langle \ell-1,i-\ell\rangle}.$$

From Section 2 it is clear that a piecewise lexsegment monomial space $W \subset S_d$ with $m_i(W) = k^{\langle \ell-1,i-\ell\rangle}\ \forall\, i = 1,\dots,\ell$ must exist. We have $V \subset W$ by Lemma 6.2, so we get also the inequality:

$$m_i(V) \leq k^{\langle \ell-1,i-\ell\rangle}.$$

$\square$

We introduce the function $\mathbb{T} : \mathbb{N}^r \to \mathbb{N}^r$ such that $\mathbb{T}(\mathbf{v}) = (v_1,\ v_1+v_2,\dots,\ v_1+v_2+\ldots+v_r)$, where $\mathbf{v} = (v_1,\dots,v_r)$. Furthermore, we define $\mathbb{S}_q(\mathbf{v})$ as the last entry of $\mathbb{T}^q(\mathbf{v})$.

Remark 6.4. The significance of the above definition is the following: Let $I \subset S = K[x_1,\dots,x_n]$ be a stable ideal generated in one degree. One can easily show that, for all $q \in \mathbb{N}$ and $i \in \{1,\dots,n\}$,

$$\mathbb{S}_q((m_1(I),m_2(I),\dots,m_i(I))) = m_i(\mathfrak{m}^q I).$$

Notice that we can also rephrase the second condition of Theorem 5.3 as

$$\mu_{i,j} \geq \mathbb{S}_1((\mu_{1,j-1},\mu_{2,j-1},\dots,\mu_{i,j-1})).$$

Example 6.5. In the next theorem the functions $\mathbb{S}_q$ will play a crucial role. Especially, using Remark 6.4, Lemma 6.3 and Remark 6.1, one has:

$$\begin{aligned}
\mathbb{S}_q((k^{\langle \ell-1,1-\ell\rangle},k^{\langle \ell-1,2-\ell\rangle},\dots,k^{\langle \ell-1,i-\ell\rangle})) &= \mathbb{S}_q((m_1(U(\ell,k,d)),m_2(U(\ell,k,d)),\dots,m_i(U(\ell,k,d))))\\
&= m_i(\mathfrak{m}^q U(\ell,k,d)))\\
&= m_i(U(\ell,m_\ell(\mathfrak{m}^q U(\ell,k,d)),d+q)))\\
&= \mathbb{S}_q((k^{\langle \ell-1,1-\ell\rangle},k^{\langle \ell-1,2-\ell\rangle},\dots,k))^{\langle \ell-1,i-\ell\rangle}.
\end{aligned}$$

Notice that the first time $\mathbb{S}_q$ is applied to a vector in $\mathbb{N}^i$, whereas the last time to a vector in $\mathbb{N}^\ell$.

Let I be a homogeneous ideal of S and $\beta_{i,j}=\beta_{i,j}(I)$ its graded Betti numbers. Let the extremal Betti numbers of I be

$$\beta_{i_1,i_1+j_1},\beta_{i_2,i_2+j_2},\ldots,\beta_{i_k,i_k+j_k}.$$

Notice that $k<n$, and up to a reordering, we can assume $0<i_1<i_2<\ldots<i_k<n$ and $j_1>j_2>\ldots>j_k\geq 0$. If I is a stable ideal then, exploiting the Eliahou-Kervaire formula, one can check that $\beta_{i,i+j}(I)$ is extremal if and only if $m_{i+1,j}(I)\neq 0$ and $m_{p+1,q}(I)=0$ for all $(p,q)\neq(i,j)$ such that $p\geq i$ and $q\geq j$. In this case, moreover, we have $\beta_{i,i+j}(I)=m_{i+1,j}(I)$. Before showing the main result of the paper, we introduce the following concept.

Definition 6.6. Let $\mathbf{i}=(i_1,\ldots,i_k)$ and $\mathbf{j}=(j_1,\ldots,j_k)$ be such that $0<i_1<i_2<\ldots<i_k<n,\ j_1>j_2>\ldots>j_k>0$. We say that $I\subset S$ is a $(\mathbf{i},\mathbf{j})$-*lex ideal* if $I=\sum_{p=1}^{k}(L_p)$, where L_p is a lexsegment ideal generated in degree j_p in $K[x_1,\ldots,x_{i_p+1}]$.

Theorem 6.7. *Let $\mathbf{i}=(i_1,\ldots,i_k)$ and $\mathbf{j}=(j_1,\ldots,j_k)$ be such that $0<i_1<i_2<\ldots<i_k<n$ and $j_1>j_2>\ldots>j_k>0$, and let $b_1,\ldots,b_k$ be positive integers. For all $p=1,\ldots,k$ let :*

$$\mathbf{v}^p=(b_p^{\langle i_p,-i_p\rangle},b_p^{\langle i_p,1-i_p\rangle},\ldots,b_p^{\langle i_p,i_{p-1}-i_p\rangle})\in\mathbb{N}^{i_{p-1}+1}.$$

If K has characteristic 0, then the following are equivalent:

- (i) *There is a homogeneous ideal $I\subset S$ with extremal Betti numbers $\beta_{i_p,i_p+j_p}(I)=b_p$ for all $p=1,\ldots,k$.*
- (ii) *There is a strongly stable ideal $I\subset S$ with extremal Betti numbers $\beta_{i_p,i_p+j_p}(I)=b_p$ for all $p=1,\ldots,k$.*
- (iii) $b_k\leq\binom{i_k+j_k-1}{i_k}$ *and* $\mathbb{S}_{j_p-j_{p+1}}(\mathbf{v}^{p+1})+b_p\leq\binom{i_p+j_p-1}{i_p}$ *for all $p=1,\ldots,k-1$.*
- (iv) *There is an $(\mathbf{i},\mathbf{j})$-lex ideal $I\subset S$ with extremal Betti numbers $\beta_{i_p,i_p+j_p}(I)=b_p$ for all $p=1,\ldots,k$.*

Proof. (i) $\iff$ (ii) follows by by [BCP, Theorem 1.6]. (iv) $\implies$ (i) is obvious.

(ii) $\implies$ (iii). By what said before the theorem, we can replace $\beta_{i_p,i_p+j_p}(I)$ by $m_{i_p+1,j_p}(I)$ with $m_{r+1,s}(I)=0$ for all $(r,s)\neq(i_p,j_p)$ such that $r\geq i_p$ and $s\geq j_p$. Since $m_{i_k+1,j_k}(I)=b_k$, we have

$$b_k\leq\binom{i_k+j_k-1}{i_k}.$$

We must have that:

$$\begin{aligned}
m_{i_{k-1}+1}\left(\mathfrak{m}^{j_{k-1}-j_k}(I_{\langle j_k\rangle})\right)+b_{k-1} &= |\{\text{monomials } u\in I_{\langle j_k\rangle}\cap S_{j_{k-1}} \text{ with } m(u)=i_{k-1}+1\}|\\
&\quad +|\{\text{monomials } u\in I_{j_{k-1}}\setminus I_{\langle j_{k-1}-1\rangle} \text{ with } m(u)=i_{k-1}+1\}|\\
&\leq |\{\text{monomials } u\in S_{j_{k-1}} \text{ with } m(u)=i_{k-1}+1\}|\\
&= \binom{i_{k-1}+j_{k-1}-1}{i_{k-1}}
\end{aligned}$$

From the discussion before the theorem, we also have:

$$m_i\left(I_{\langle j_k\rangle}\right)\geq b_k^{\langle i_k,i-i_k-1\rangle}\quad\forall\, i\leq i_k.$$

We eventually get:

$$m_{i_{k-1}+1}\left(\mathfrak{m}^{j_{k-1}-j_k}(I_{\langle j_k\rangle})\right)\geq\mathbb{S}_{j_{k-1}-j_k}(\mathbf{v}^k).$$

Putting together the above inequalities we obtain, for $p=k-1$,

$$\mathbb{S}_{j_p-j_{p+1}}(\mathbf{v}^{p+1})+b_p\leq\binom{i_p+j_p-1}{i_p},$$

and we can go on in the same way to show this for all $p=1,\ldots,k-1$.

(iii) $\implies$ (iv). If $b_k\leq\binom{i_k+j_k-1}{i_k}$, then we can form $U(i_k+1,b_k,j_k)$. Let us call ${}^kI=U(i_k+1,b_k,j_k)$. We have that:

$$m_{i_{k-1}+1}\left(({}^kI)_{\langle j_{k-1}\rangle}\right)=\mathbb{S}_{j_{k-1}-j_k}(\mathbf{v}^k).$$

From Remark 6.1, we deduce that

$$({}^kI)_{\langle j_{k-1}\rangle}\cap K[x_1,\ldots,x_{i_{k-1}+1}]=U(i_{k-1}+1,\mathbb{S}_{j_{k-1}-j_k}(\mathbf{v}^k),j_{k-1})\cap K[x_1,\ldots,x_{i_{k-1}+1}].$$

By the assumed numerical conditions, $U(i_{k-1}+1, \mathbb{S}_{j_{k-1}-j_k}(\mathbf{v}^k)+b_{k-1}, j_{k-1})$ exists and contains exactly b_{k-1} new monomials u such that $m(u) = i_{k-1}+1$. Therefore set:

$$^{k-1}I' = (U(i_{k-1}+1, \mathbb{S}_{j_{k-1}-j_k}(\mathbf{v}^k)+b_{k-1}, j_{k-1})).$$

and

$$^{k-1}I = {}^kI + {}^{k-1}I'.$$

By construction ^{k-1}I is a $((i_{k-1}, i_k), (j_{k-1}, j_k))$-lex ideal with extremal Betti numbers $\beta_{i_{k-1}, i_{k-1}+j_{k-1}}(^{k-1}I) = b_{k-1}$ and $\beta_{i_k, i_k+j_k}(^{k-1}I) = b_k$. Keeping on with the recursion we will end up with the desired $(\mathbf{i}, \mathbf{j})$-lex ideal $I = {}^1I$. □

Remark 6.8. For the reader who likes more the language of algebraic geometry, Theorem 6.7 can be used in the following setting: Let $X \subset \mathbb{P}^{n-1}$ be a projective scheme over a field of characteristic 0 and $\mathscr{I}_X$ its ideal sheaf. Then, by the graded version of the Grothendieck's local duality, $\beta_{i,i+d}$ is an extremal Betti number of the ideal $\bigoplus_{m\in\mathbb{N}} \Gamma(X, \mathscr{I}_X(m)) \subset S$ if and only if, setting $p = n-i-1$ and $q = d-1$:

(1) $p \geq 1$;
(2) $\dim_K(H^p(X, \mathscr{I}_X(q-p))) = \beta_{i,i+d} \neq 0$.
(3) $H^r(X, \mathscr{I}_X(s-r)) = 0$ for all $(r,s) \neq (p,q)$ with $1 \leq r \leq p$ and $s \geq q$.

Example 6.9. Let us consider the following Betti table:

$$\begin{pmatrix} * & * & * & * & * & * & \cdots \\ * & * & * & a & 0 & 0 & \cdots \\ * & * & * & 0 & 0 & 0 & \cdots \\ * & * & b & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}.$$

Theorem 6.7 implies that there exists a homogeneous ideal in a polynomial ring (of characteristic 0) whose Betti table looks like the above one (where a and b are extremal) if and only if we are in one of the following cases:

(i) $a = 2$ and $b = 1, 2$;
(ii) $a = 1$ and $b = 1, 2, 3, 4$.

In fact, we have $b = \beta_{2,6}$ and $a = \beta_{3,5}$. Theorem 6.7 implies $a \leq 4$.

If $a = 2$, then the vector $\mathbf{v}^2 \in \mathbb{N}^3$ is:

$$\mathbf{v}^2 = (1, 2, 2).$$

Therefore $\mathbb{S}_2(\mathbf{v}^2) = 8$, and Theorem 6.7 gives $8 + b \leq 10$. So we get $b = 1, 2$ as desired.

If $a = 1$, then the vector $\mathbf{v}^2 \in \mathbb{N}^3$ is:

$$\mathbf{v}^2 = (1, 1, 1).$$

So $\mathbb{S}_2(\mathbf{v}^2) = 6$, and Theorem 6.7 yields $b = 1, 2, 3, 4$ as desired.

Eventually, if $a > 2$, a positive integer b satisfying the conditions of Theorem 6.7 does not exist.

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FACHBEREICH MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, CAMPUS ESSEN, 45117 ESSEN, GERMANY
E-mail address: juergen.herzog@uni-essen.de

DEPARTMENT OF MATHEMATICS, HAKIM SABZEVARI UNIVERSITY, SABZEVAR, IRAN
E-mail address: leila-sharifan@aut.ac.ir

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI GENOVA, ITALY
E-mail address: varbaro@dima.unige.it