F-splittings of the polynomial ring and compatibly split homogeneous ideals

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Frobenius map

Let R be a Noetherian ring of prime characteristic p. The **Frobenius map** on R is the ring homomorphism

$$F: \quad R \longrightarrow R. \\ r \mapsto r^p$$

Denote by F_*R the R-module obtained restricting the scalars:

- $F_*R = R$ as additive group;
- $r \cdot x = r^p x$ for all $r \in R$ and $x \in F_*R$.

In this way we can think at F as the map of R-modules $F : R \longrightarrow F_*R$ sending $r \in R$ to $r^p \in F_*R = R$.

F-pure and F-split rings

Definition

R is F-**pure** if $F : R \to F_*R$ is a pure map of R-modules. That is, $F \otimes 1_M : M \to F_*R \otimes_R M$ is injective for any R-module M.

Remark/Exercise

- If R is F-pure, then it is reduced.
- If R is regular, then it is F-pure (since F_*R is a faithfully flat R-module by Kunz's theorem).

Definition

R is F-**split** if $F : R \to F_*R$ is a split-inclusion of R-modules. In this case, a map $\theta \in \mathrm{Hom}_R(F_*R, R)$ such that $\theta \circ F = 1_R$ is called an *F-splitting* of R.

Remark/Exercise

If R is F-split, then it is F-pure.

Compatibly split ideals

Assume R is F-split, and fix an F-splitting $\theta \in \mathrm{Hom}_R(F_*R, R)$. We say that an ideal $I \subset R$ is **compatibly split** (with respect to θ) if $\theta(I) \subset I$.

Remark

Of course $I \subset \theta(I)$ for any ideal $I \subset R$. In fact, if $r \in I$, then $r^p \in I$ and $\theta(r^p) = \theta(F(r)) = r$. In particular, an ideal $I \subset R$ is compatibly split if and only if $I = \theta(I)$.

If I is compatibly split, then $\overline{\theta} : F_*(R/I) = (F_*R)/I \to R/I$ is a well-defined map of Abelian groups. It is straightforward to check that it is also a map of R/I-modules. Hence $\overline{\theta}$ is an F-splitting of R/I, and then R/I is F-split.

Compatibly split ideals

Proposition

Let R be F-split, and fix an F-splitting $\theta \in \mathrm{Hom}_R(F_* R, R)$.

1. If $I \subset R$ is a compatibly split ideal, so is $I : J$ for all $J \subset R$.
2. If $I, J \subset R$ are compatibly split ideals, so are $I \cap J$ and $I + J$.

Proof. Let us prove (1): If $r \in \theta(I : J)$ and $x \in J$, then $r = \theta(y)$ where xy, and so $x^p y$, belongs to I. So $xr = x\theta(y) = \theta(x \cdot y) = \theta(x^p y) \in I$. Point (2) is straightforward to show. □

Example

Let K be a field of positive characteristic. The reduced ring $R = K[X, Y]/(X(X + Y)Y)$ is not F-split: in fact, if there was an F-splitting of R, (0) would be a compatibly split ideal, and so $(x) = (0) : (x + y)y$, $(x + y) = (0) : xy$, $(y) = (0) : x(x + y)$ and $(x(x + y)) = (x) \cap (x + y)$ would be compatibly split (where $x = \overline{X}$ and $y = \overline{Y}$). Hence the ideal $I = (x(x + y), y) = (x^2, y)$ would be compatibly split, whereas it is not even radical.

F-pure and F-split rings

Even if there are rings R that are F-pure but not F-split, in some context the two concepts are the same:

Proposition

Assume that R satisfyies one of the following conditions:

1. F_*R is a finitely generated R-module.
2. $R = \bigoplus_{i\in\mathbb{Z}} R_i$ is a graded ring having a unique homogeneous ideal $\mathfrak{m}$ which is maximal by inclusion and such that R_0 is a complete local ring.

Then R is F-pure if and only if it is F-split. In (2), these two properties are furthermore equivalent to the injectivity of the map $F \otimes 1_E : E \to M \otimes_R E$ where E is the injective hull of $R/\mathfrak{m}$.

Since we will work with Noetherian $\mathbb{N}$-graded rings $R = \bigoplus_{i\in\mathbb{N}} R_i$ with R_0 a field, which automatically satisfy (2), for this seminar we are allowed to confuse the F-pure and F-split notions.

F-splitting of the polynomial ring

From now on K is a perfect field of positive characteristic p, and $R = K[X_1, \dots, X_n]$ the polynomial ring in n variables over K.

Remark/Exercise

$F_* R$ is a free R-module of rank p^n. An R-basis is given by the set

$$\{X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} : \ 0 \leq a_i < p \ \ \forall \ i = 1, \dots, n\}.$$

In particular, R is F-split.

We want to study the F-splittings of R. Of course $\mathrm{Hom}_R(F_* R, R)$ is a free R-module of rank p^n and R-basis dual to the one of the above remark, say $\{\phi_{a_1, a_2, \dots, a_n} : \ 0 \leq a_i < p \ \ \forall \ i = 1, \dots, n\}$.

Furthermore, given $f_{a_1, a_2, \dots, a_n} \in R$, the element

$$\theta = \sum_{0 \leq a_i < p} f_{a_1, a_2, \dots, a_n} \cdot \phi_{a_1, a_2, \dots, a_n} \in \mathrm{Hom}_R(F_* R, R)$$

is an F-splitting of R if and only if $f_{0,0,\dots,0} = 1$.

F-splitting of the polynomial ring

Recall that F_*R, beyond being an R-module, is a ring (isomorphic to R), so it makes sense to consider the following F_*R-module structure on $\mathrm{Hom}_R(F_*R, R)$:

$$\begin{aligned} f \star \theta : F_*R &\to R \qquad\qquad \forall\, f \in F_*R, \theta \in \mathrm{Hom}_R(F_*R, R) \\ g &\mapsto \theta(fg) \end{aligned}$$

To understand the F_*R-module structure of $\mathrm{Hom}_R(F_*R, R)$, call $\mathrm{Tr} := \phi_{p-1,p-1,\ldots,p-1}$. Then the following is an isomorphism of F_*R-modules:

$$\begin{aligned} \Psi : \quad F_*R \to &\quad \mathrm{Hom}_R(F_*R, R) \\ f \mapsto &\quad f \star \mathrm{Tr} \end{aligned}$$

The fact that Ψ is an injective map of F_*R-modules is clear. For the surjectivity, just notice that, if $a_1, \ldots, a_n$ are such that $0 \leq a_i < p$ for all i, we have $\phi_{a_1,\ldots,a_n} = X_1^{p-a_1-1} \cdots X_n^{p-a_n-1} \star \mathrm{Tr}$.

F-splitting of the polynomial ring

So, there is a 1-1 correspondence between polynomials of R and elements of $\mathrm{Hom}_R(F_*R, R)$, given by $f \leftrightarrow f \star \mathrm{Tr}$. Furthermore, $f \star \mathrm{Tr}$ is an F-splitting of R if and only if the following two conditions hold true simultaneously:

1. $X_1^{p-1} \cdots X_n^{p-1} \in \mathrm{supp}(f)$ and its coefficient in f is 1.
2. If $X_1^{u_1} \cdots X_n^{u_n} \in \mathrm{supp}(f)$ and $u_1 \equiv \ldots \equiv u_n \equiv -1 \pmod{p}$, then $u_i = p - 1 \;\forall\; i = 1, \ldots, n$.

Remark/Exercise

Notice that, if R is equipped with a positive grading, (i.e. $\deg(X_i)$ is a positive integer for all $i = 1, \ldots, n$), then the second condition above is automatic whenever f is a homogeneous polynomial satisfying (1).

F-splitting of the polynomial ring

Definition

If $I \subset R$ is an ideal, then $I^{[p]} = (f^p : f \in I)$. Equivalently, $I^{[p]} = (f^p : f \in \mathcal{I})$ whenever $I = (\mathcal{I})$.

Proposition

For any $f \in R$ and any ideal $I \subset R$, we have:

$$(f \star \mathrm{Tr})(I) \subset I \Leftrightarrow f \in I^{[p]} : I.$$

Therefore, given an ideal $I \subset R$, if we can find a polynomial $f \in I^{[p]} : I$ such that

1. $X_1^{p-1} \cdots X_n^{p-1} \in \mathrm{supp}(f)$ and its coefficient in f is 1;
2. If $X_1^{u_1} \cdots X_n^{u_n} \in \mathrm{supp}(f)$ and $u_1 \equiv \ldots \equiv u_n \equiv -1 \pmod{p}$, then $u_i = p - 1 \ \forall\ i = 1, \ldots, n$,

then $f \star \mathrm{Tr}$ is an F-splitting of R for which I is compatibly split. In particular R/I is F-split.

F-splitting of the polynomial ring

Theorem (Fedder's criterion)

Equip R with a positive grading, and consider a homogeneous ideal $I \subset R$. The following facts are equivalent:

1. R/I is F-split.
2. There is an F-splitting of R for which I is compatibly split.
3. $I^{[p]} : I \not\subset (X_1^p, \ldots, X_n^p)$.

The third condition above can be rephrased like this: there is $f \in I^{[p]} : I$ such that $X_1^{p-1} \cdots X_n^{p-1} \in \mathrm{supp}(f)$ and its coefficient in f is 1; in this case I is compatibly split with respect to the F-splitting $f \star \mathrm{Tr}$. What if, furthermore, there is a monomial order such that

$$\mathrm{in}(f) = X_1^{p-1} \cdots X_n^{p-1} \quad ???$$

F-splitting of the polynomial ring

First of all, let us see what are the compatibly split ideals of the F-splitting of R

$$\theta = X_1^{p-1} \cdots X_n^{p-1} \star \mathrm{Tr} \in \mathrm{Hom}_R(F_* R, R).$$

Proposition

The compatibly split ideals w.r.t. θ are precisely the squarefree monomial ideals of R.

Proof. Let $I \subset R$ be an ideal generated by squarefree monomials $u_1, \ldots, u_k \in R$. As an R-submodule of $F_* R$, I is generated by

$$\{u_j X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} : \ j = 1, \ldots, k, \ 0 \leq a_i < p \ \ \forall \ i = 1, \ldots, n\}.$$

Note that $\theta(u_j X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}) \neq 0$ iff $X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} = u_j^{p-1}$, so:

$$\theta(u_j X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}) = \theta(u_j^p) = \theta(u_j \cdot 1) = u_j \theta(1) = u_j \in I.$$

F-splitting of the polynomial ring

Check as an exercise that a compatibly split ideal w.r.t. θ must be a monomial ideal. $\square$

Proposition/Exercise (Knutson)

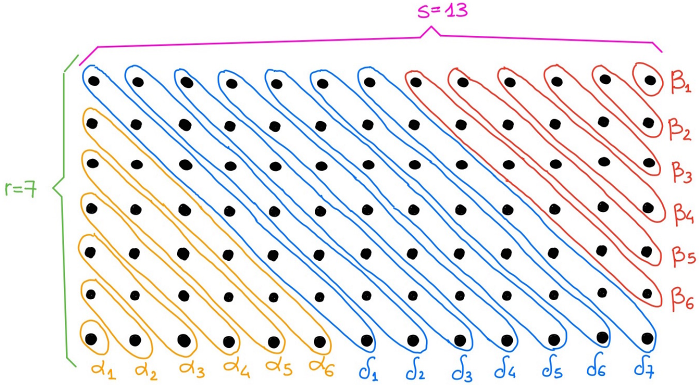
Let $<$ be a monomial order on R. Then, for any $g \in R$, either $\mathrm{Tr}(\mathrm{in}(g)) = 0$ or $\mathrm{Tr}(\mathrm{in}(g)) = \mathrm{in}(\mathrm{Tr}(g))$.

Corollary

Let $<$ be a monomial order on R. If $I \subset R$ is an ideal such that there is $f \in I^{[p]} : I$ with, $\mathrm{in}(f) = X_1^{p-1} \cdots X_n^{p-1}$, then R/I is F-split and $\mathrm{in}(I)$ is a squarefree monomial ideal.

The ideal of maximal minors of a generic matrix

Let $X = (X_{ij})$ be a $r \times s$ generic matrix, and suppose $r \leq s$. Let $R = K[X]$ and $I \subset R$ the ideal generated by the maximal minors of X. Then I is a prime ideal of height $s - r + 1$, and contains the complete intersection $C = (\delta_1, \ldots, \delta_{s-r+1}) \subset R$, where the δ_i's, as well as the α_j's and the β_j's, where j runs from 1 to $r-1$, are the minors whose main diagonals are illustrated in the picture below.



The ideal of maximal minors of a generic matrix

Put $\Delta = \prod_{i=1}^{r-1} \alpha_i \prod_{i=1}^{s-r+1} \delta_i \prod_{i=1}^{r-1} \beta_i \in R$, and notice that

$$\mu^{p-1} \in \operatorname{supp}(\Delta^{p-1}), \quad \text{where } \mu = \prod_{i=1}^{r} \prod_{j=1}^{s} X_{ij} :$$

indeed, if $<$ is the lexicographic term order with

$$X_{11} > X_{12} > \ldots > X_{1s} > X_{21} > \ldots > X_{2s} > \ldots > X_{rs},$$

then $\text{in}(\Delta) = \mu$, so that $\text{in}(\Delta^{p-1}) = \text{in}(\Delta)^{p-1} = \mu^{p-1}$.

Therefore $\theta = \Delta^{p-1} \star \mathrm{Tr}$ is an F-splitting of R, and since $C\Delta^{p-1} \subset C^{[p]}$, the ideal C is compatibly split w.r.t. θ. Since I is a prime ideal containing C and $\mathrm{ht}\, I = \mathrm{ht}\, C$, then $I = C : f$ for some $f \in R$. So I is a compatibly split ideal w.r.t. θ.

Indeed, one can show that, for any positive integer $t \leq r$, the ideal I_t generated by the t-minors of X is compatibly split w.r.t. θ, using a result of De Concini, Eisenbud and Process stating that

$$\Delta \in I_t^{(\mathrm{ht}\, I_t)}.$$

Recently, Lisa Seccia even proved that any I_t can be obtained by iteratively taking colons and sums starting from Δ, as we did for $I = I_r$.

Anyway, by what previously said, we get that $\mathrm{in}(I)$ is a squarefree monomial ideal. This is an important information, for example we can deduce from a result of Conca and myself that

$$\mathrm{reg}(\mathrm{in}(I)) = \mathrm{reg}(I) = r,$$

where the second equality holds since the Eagon-Northcott complex resolves I.

The ideal of maximal minors of a generic matrix

If $1 \leq i_1 < \ldots < i_r \leq s$, denoting by $[i_1 \ldots i_r]$ the r-minor of X insisting on the columns $i_1, \ldots, i_r$, one has

$$\mathrm{in}([i_1 \ldots i_r]) = X_{1i_1} \cdots X_{ri_r},$$

so $\{[i_1 \ldots i_r] : 1 \leq i_1 < \ldots < i_r \leq s\}$ is a Gröbner basis of I, because

1. $I = ([i_1 \ldots i_r] : 1 \leq i_1 < \ldots < i_r \leq s)$.
2. $\mathrm{in}([i_1 \ldots i_r]) \neq \mathrm{in}([j_1 \ldots j_r])$ if $[i_1 \ldots i_r] \neq [j_1 \ldots j_r]$.
3. $\mathrm{in}(I)$ has a linear resolution.

All of this was already known, but the same argument gives the following...

A general statement

Proposition

Let $I \subset R = K[X_1, \ldots, X_n]$ be a homogeneous ideal which is compatibly split w.r.t. $f \star \mathrm{Tr}$ with $\mathrm{in}(f) = X_1^{p-1} \cdots X_n^{p-1}$.
If I has a linear resolution, then there exists a minimal system of generators of I which is a Gröbner basis.

An open question about the ideal of maximal minors of a generic matrix

We conclude stating a problem about maximal minors. Whenever $1 \leq i_1 < \ldots < i_r \leq s$, from the previous discussion, it follows that there is a writing:

$$\Delta^{p-1}[i_1 \ldots i_r] = \sum_{1 \leq j_1 < \ldots < j_r \leq s} f_{j_1,\ldots,j_r}^{i_1,\ldots,i_r} [j_1 \ldots j_r]^p, \quad f_{j_1,\ldots,j_r}^{i_1,\ldots,i_r} \in R.$$

It is not known, at least to my knowledge, an explicit writing like above. It would be interesting to know one minimizing the set

$$\{1 \leq j_1 < \ldots < j_r \leq s : f_{j_1,\ldots,j_r}^{i_1,\ldots,i_r} \neq 0\}.$$

Thank you for the attention!