Test and multiplier ideals

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The multiplicity is a quite rough measurement of singularities though...

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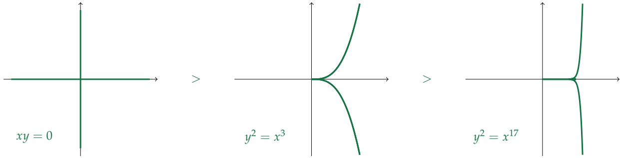

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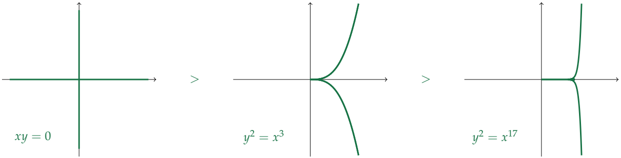

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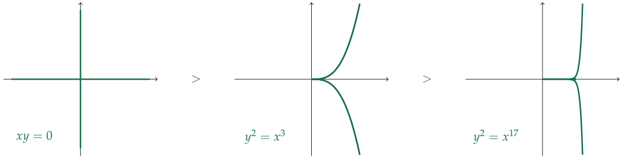

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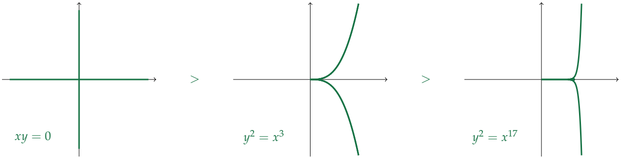

The three curves above have multiplicity 2 at the origin. However, the above singularities are evidently quite different. For today, we will consider the first singularity better than the second, which in turns will be better than the third...

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$$\int_B \frac{1}{|f|^2}$$

does not converge for any neighborhood B of 0.

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converges for a small neighborhood B of 0 iff $\lambda < \min_i\{1/a_i\}$. In particular, if f is a square-free monomial, then the above integral converges for any $\lambda < 1$, as in the smooth case!

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Def.: The log-canonical threshold of $f \in \mathfrak{m} = (x_1, \ldots, x_N)$ is

$$\mathrm{lct}(f) = \sup\{\lambda \in \mathbb{R}_{>0} : \exists \text{ a neighborfood } B \text{ of } 0 \text{ s.t. } \int_B \frac{1}{|f|^{2\lambda}} < \infty\}.$$

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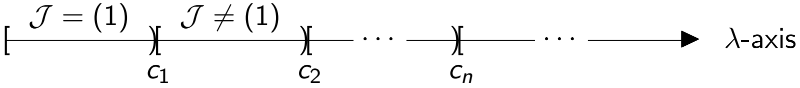

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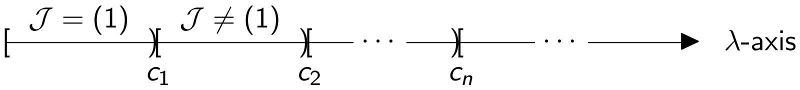

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The c_i above are called jumping numbers. Note that $c_1 = \mathrm{lct}(f)$.

Multiplier ideals in general

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Even more generally, one can define the multiplier ideals $\mathcal{J}(\lambda \bullet I)$ (and so the jumping numbers and the log-canonical threshold) for any ideal $I = (f_1, \ldots, f_r) \subseteq \mathfrak{m}$, and not just for a polynomial:

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There is also a way to define the multiplier ideals via resolution of singularities (over a field of characteristic 0). Although today we will not discuss this perspective, it is a fundamental point of view, providing several techniques to study multiplier ideals. Furthermore, properties that are not clear from the analytic approach suddenly become evident from the algebro-geometric one: e.g., that the log-canonical threshold (indeed all the jumping numbers) is a rational number!

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What are the multiplier ideals of the G-invariant ones?

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- $V = \bigwedge^2 E$ and $G = \mathrm{GL}(E)$ (the G-invariant ideals are ideals of pfaffians of a generic skew-symmetric matrix, their products, their symbolic powers, sums of these, and more...)

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In the final part of the talk we will introduce the F-pure threshold, that is the characteristic-p-analog of the log-canonical threshold.

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Notice that this definition does not depend on the choice of generators because $\mathrm{char}(K) = p$!

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So $\{\nu(e)/p^e\}_{e \in \mathbb{N}}$ is a monotone sequence in $[0, 1]$, thus it admits a limit. The F-pure threshold of f is:

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Recall that, by a classical result of Elkies, E_p is ordinary for infinitely many primes p as well it is supersingular for infinitely primes p.

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Hara-Yoshida: $\lim_{p\to\infty} \mathrm{fpt}(g_p) = \mathrm{lct}(g_0)$.

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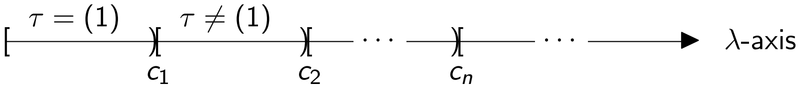

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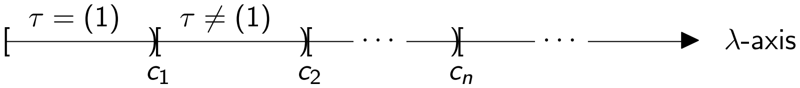

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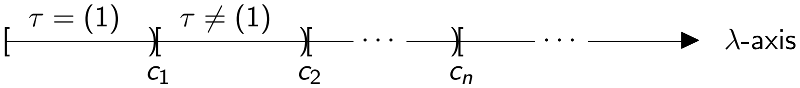

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Blickle-Mustață-Smith: In our setting, all the F-jumping numbers (so in particular the F-pure threshold) are rational numbers.

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