Commutative algebra and Coxeter groups

Matteo Varbaro (University of Genoa, Italy)

The 38th Symposium on Commutative Algebra in Japan
Hayama, 19/11/2016

Motivations from commutative algebra

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, and $I \subseteq S$ be a **quadratic** ideal (i.e. generated by quadrics).

The **minimal graded free resolution** of S/I has the form:

$$0 \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{k,j}} \to \dots \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{2,j}} \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{1,j}} \to S \to S/I$$

where k is the projective dimension of S/I, $\beta_{1,j} = 0$ if $j \neq 2$, and $\beta_{1,2}$ is the number of minimal generators of I. The **Castelnuovo-Mumford regularity** of S/I is

$$\operatorname{reg} S/I = \max\{j - i : \beta_{i,j} \neq 0\}.$$

Mayr-Meyer, 1982

There exist quadratic ideals $I \subseteq S$ for which $\operatorname{reg} S/I$ is doubly exponential in n.

Motivations from commutative algebra

The ideals of Mayr-Meyer have the property that already the first syzygy module has minimal generators in a very high degree, indeed in their examples $\beta_{2,j} \neq 0$ for a certain $j > 2^{2^{n/10}}$.

If the first syzygy module of I is linearly generated (i.e. $\beta_{2,j} = 0$ whenever $j \neq 3$), no example with $\operatorname{reg} S/I > n$ is known.

Definition

We say that S/I **satisfies the property** N_p if $\beta_{i,j} = 0$ for all $i \leq p$, $j > i + 1$. The **Green-Lazarsfeld index** of S/I is

$$\operatorname{index} S/I = \sup\{p \in \mathbb{N} : S/I \text{ satisfies the property } N_p\}$$

For example, S/I satisfies N_1 just means that I is generated by quadrics, S/I satisfies N_2 means that I is generated by quadrics and has first linear syzygies .....

For this talk we will focus in the case $I = I_\Delta \subseteq S = K[x_1, \ldots, x_n]$ is a square-free monomial ideal (where Δ is a simplicial complex on n vertices). In this case S/I_Δ is denoted by $K[\Delta]$ and called the **Stanley-Reisner ring** of Δ.

Dao, Huneke, Schweigh, 2013

If $K[\Delta]$ satisfies the property N_p for some $p \geq 2$, then

$$\operatorname{reg} K[\Delta] \leq \log_{\frac{p+3}{2}} \left(\frac{n-1}{p} \right) + 2$$

Fixed $p \geq 2$, the following is thus a natural question:

Question A_p

Is there a global bound for $\operatorname{reg} K[\Delta]$ if $K[\Delta]$ satisfies N_p?

Let Γ be a **simple graph** on a (possibly infinite) vertex set V. Given two vertices $v, w \in V$, a **path** e from v to w consists in a subset of vertices

$$\{v = v_0, v_1, v_2, \dots, v_k = w\}$$

such that $\{v_i, v_{i+1}\}$ is an edge for all $i = 0, \dots, k-1$. The **length** of such a path is $\ell(e) = k$. The **distance** between v and w is

$$d(v, w) := \inf\{\ell(e) : e \text{ is a path from } v \text{ to } w\}.$$

A path e from v to w is called a **geodesic path** if $\ell(e) = d(v, w)$. A **geodesic triangle** of vertices v_1, v_2 and v_3 consists in three geodesic paths e_i from v_i to v_{i+1} (mod 3) for $i = 1, 2, 3$.

Metric group theory

For $\delta > 0$, a geodesic triangle e_1, e_2, e_3 is δ-**slim** if $d(v, e_i \cup e_j) \leq \delta$ for all $v \in e_k$ and $\{i, j, k\} = \{1, 2, 3\}$. The graph Γ is **hyperbolic** if there exists $\delta > 0$ such that each geodesic triangle of Γ is δ-slim.

Let G be a **group** and $\mathcal{S}$ a set of (distinct) generators of G (not containing the identity). The **Cayley graph** $\text{Cay}(G, \mathcal{S})$ is the simple graph with:

- G as vertex set;
- as edges, the sets $\{g, gs\}$ where $g \in G$ and $s \in \mathcal{S}$.

Gromov

Given two finite sets of generators $\mathcal{S}$ and $\mathcal{S}'$ of G, $\text{Cay}(G, \mathcal{S})$ is hyperbolic if and only if $\text{Cay}(G, \mathcal{S}')$ is.

Definition

A group G is **hyperbolic** if it has a finite set of generators $\mathcal{S}$ such that $\mathrm{Cay}(G, \mathcal{S})$ is hyperbolic.

For example, $\mathbb{Z}^2$ is not hyperbolic: choosing $\mathcal{S} = \{(1,0),(0,1)\}$,

- $e_1 = \{(0,0),(0,1),(0,2),\ldots,(0,n)\}$;
- $e_2 = \{(0,0),(1,0),(2,0),\ldots,(n,0)\}$;
- $e_3 = \{(0,n),(1,n),\ldots,(n,n),(n,n-1),(n,n-2),\ldots,(n,0)\}$.

The geodesic paths e_1, e_2, e_3 form a geodesic triangle with vertices $(0,0),(0,n),(n,0)$ in $\mathrm{Cay}(\mathbb{Z}^2, \mathcal{S})$, but $d((n,n), e_1 \cup e_2) = n$. By Gromov's result, $\mathrm{Cay}(\mathbb{Z}^2, \mathcal{S}')$ is not hyperbolic for any finite set of generators $\mathcal{S}'$ of $\mathbb{Z}^2$, therefore $\mathbb{Z}^2$ is not hyperbolic.

Virtual cohomological dimension

The **cohomological dimension** of a group G is defined as:

$$\text{cd}\, G = \sup\{n \in \mathbb{N} : H^n(G; M) \neq 0 \ \text{ for some } \mathbb{Z}G\text{-module } M\}.$$

If G has nontrivial torsion, then it is well known that $\text{cd}\, G = \infty$.

A group G is **virtually torsion-free** if it has a finite index subgroup which is torsion-free. By a result of Serre, if Γ and Γ' are two finite index torsion-free subgroups of G, then

$$\text{cd}\, \Gamma = \text{cd}\, \Gamma'.$$

So it is well-defined the **virtual cohomological dimension** of a virtually torsion-free group G:

$\text{vcd}\, G = \text{cd}\, \Gamma$ where Γ is a finite index torsion-free subgroup of G.

Coxeter groups

A **Coxeter group** is a pair $(G, \mathcal{S})$ where G is a group with a presentation of the type $\langle \mathcal{S} \mid \mathcal{R} \rangle$ such that:

- $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ is a system of generators of G;
- the relations $\mathcal{R}$ are of the form $(s_i s_j)^{m_{ij}} = e$ where $m_{ii} = 1$ for all $i = 1, \ldots, n$ and $m_{ij} \in \{2, 3, \ldots\} \cup \{\infty\}$ otherwise.

A Coxeter group is **right-angled** if and only if $m_{ij} \in \{1, 2, \infty\}$.

Remark

For $i \neq j$, notice that $m_{ij} = 2$ if and only if $s_i s_j = s_j s_i$.

A Coxeter group $(G, \mathcal{S})$ can be embedded in $\mathrm{GL}_n(\mathbb{C})$ (where $n = |\mathcal{S}|$). So, by Selberg's lemma, a Coxeter group is virtually torsion-free; in particular the virtual cohomological dimension of a Coxeter group is well-defined .....

Question B (Gromov)

Is there a global bound for the virtual cohomological dimension of a right-angled hyperbolic Coxeter group?

Constantinescu, Kahle, _ ,2016

Questions A_2 and B are equivalent.

In particular, since Gromov's question has been negatively answered by Januszkiewicz and Świątkowski, we get:

Corollary

For any $r \in \mathbb{N}$, there exists a simplicial complex Δ such that $K[\Delta]$ satisfies N_2 and $\operatorname{reg} K[\Delta] \geq r$.

The connection

Let $(G, \mathcal{S})$ be a Coxeter group. A subset of $\mathcal{S}$ is called **spherical** if the subgroup of G it generates is finite. The **nerve** $\mathcal{N}(G)$ of $(G, \mathcal{S})$ is the (finite) simplicial complex with vertex set $\mathcal{S}$ and the spherical sets as faces. We proved:

Constantinescu, Kahle, _ ,2016

$$\operatorname{vcd} G = \max\{\operatorname{reg} K[\mathcal{N}(G)] : K \text{ is a field}\}.$$

Notice also that, if $(G, \mathcal{S})$ is right-angled, then $\mathcal{N}(G)$ is flag. It remains thus to translate the hyperbolicity of G into some combinatorial property of $\mathcal{N}(G)$ .....

The connection

..... We already saw that the group $\mathbb{Z}^2$ is not hyperbolic. Therefore any group containing $\mathbb{Z}^2$ as a subgroup cannot be hyperbolic. For Coxeter groups the condition of not containing $\mathbb{Z}^2$ is also sufficient for being hyperbolic!

Moussong

Let $(G, \mathcal{S})$ be a Coxeter group. TFAE:

- G is hyperbolic;
- $\mathbb{Z}^2 \not\subseteq G$.

If G is furthermore right-angled, then the above conditions are equivalent to:

- $\mathcal{N}(G)$ has no induced 4-cycles.

The connection

So we get as a consequence the following:

Corollary

If $(G, \mathcal{S})$ is a right-angled Coxeter group, TFAE:

- G is hyperbolic;
- $K[\mathcal{N}(G)]$ satisfies N_2.

....

Following a construction by Osajda, we are able to negatively answer question A_p in general:

Constantinescu, Kahle, _ ,2016

For any $p \geq 2$ and any $r \in \mathbb{N}$, there exists a simplicial complex Δ such that $K[\Delta]$ satisfies N_p and $\operatorname{reg} K[\Delta] = r$.

Strategy for the proof

First of all, notice that, if Δ is the $(p+3)$-cycle, then

$$\text{index}\, K[\Delta] = p \quad \text{and} \quad \text{reg}\, K[\Delta] = 2.$$

The strategy is to induct upon the regularity and use the following:

Constantinescu, Kahle, _ ,2016

If Δ is a simplicial complex such that $\text{reg}\, K[\Delta] = r > 1$ (for any field K) and $\text{index}\, K[\Delta] = p$, then there exists a simplicial complex Γ such that:

1. $\text{reg}\, K[\Gamma] = r + 1$;
2. $\text{index}\, K[\Gamma] = p$;
3. $\text{lk}_\Gamma\, v$ is the face complex of Δ for any vertex Γ of v.

Discussion on face complex

Given a simplicial complex Δ, its **face complex** $\text{Face}(\Delta)$ is the simplicial complex whose vertices are the faces of Δ, and such that

$$\{\sigma_1, \ldots, \sigma_k\} \in \text{Face}(\Delta) \iff \bigcup_{i=1}^{k} \sigma_i \in \Delta.$$

It's easy to see that $\text{Face}(\Delta)$ is homotopically equivalent to Δ. Less easily, one can also show that

$$\text{reg}\, K[\text{Face}(\Delta)] = \text{reg}\, K[\Delta].$$

It would be nice to replace condition (3) in the previous result with

(3') $\text{lk}_\Gamma\, v = \Delta$ for any vertex Γ of v.

However for the moment we have no idea how to do .....

Discussion on the number of vertices

In the previous result, starting from a simplicial complex Δ on n vertices we construct another simplicial complex Γ on $n(p,r)$ vertices. In our construction, $n(p,r)$ is a huge number:

$$n(p,r) \sim 3^{p(2\uparrow\uparrow r)n^2}.$$

The result of Dao, Huneke and Schweigh mentioned at the beginning implies that the number $n(p,r)$ cannot be too small, however their bound is much smaller compared to our example. So there is room to improve our construction, or to sharpen the result of Dao, Huneke and Schweigh, or both .....

An open problem

For any $r \in \mathbb{N}$, our method gives a simplicial complex Δ such that $K[\Delta]$ satisfies N_2 and $\operatorname{reg} K[\Delta] = r$. However, $K[\Delta]$ is far from being Cohen-Macaulay. On the other hand we have the following:

Constantinescu, Kahle, _ ,2014

If Δ is a simplicial complex such that $K[\Delta]$ satisfies N_2 **and is Gorenstein**, then

$$\operatorname{reg} K[\Delta] \leq 4.$$

So the following arises naturally:

Question

Is there a global bound for $\operatorname{reg} K[\Delta]$ if $K[\Delta]$ satisfies N_2 **and is Cohen-Macaulay**?

ARIGATO !!!