Singularities, Serre conditions and h-vectors

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Notation

Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a standard graded algebra over a field $R_0 = K$. The *Hilbert series* of R is

$$H_R(t) = \sum_{i \in \mathbb{N}} (\dim_K R_i) t^i \in \mathbb{N}[[t]].$$

If $d = \dim R$, Hilbert proved that

$$H_R(t) = \frac{h(t)}{(1-t)^d}$$

where $h(t) = h_0 + h_1 t + h_2 t^2 + \ldots + h_s t^s \in \mathbb{Z}[t]$ is the *h-polynomial* of R. We will name the coefficients vector $(h_0, h_1, h_2, \ldots, h_s)$ the *h-vector* of R.

Motivations

Let $X = \text{Proj}\, R$. If $\dim_K R_1 = n+1$, R is the coordinate ring of the embedding $X \subset \mathbb{P}^n$, whose *degree* is $h(1) = \sum_{i \geq 0} h_i$ (also called the *multiplicity* of R). So the sum of the h_i is positive, but it can happen that some of the h_i is negative.

If R is Cohen-Macaulay, then it is easy to see that $h_i \geq 0$ for all $i \geq 0$, however without the CM assumption things get complicated. In this talk I want to discuss conditions on R and/or on X which ensure at least the nonnegativity of the first h_i's and that the degree of $X \subset \mathbb{P}^n$ is bounded below by their sum.

Motivations

Related to the kind of issues we are going to discuss is also the following classical theorem in algebraic geometry, essentially due to Del Pezzo and Bertini:

Theorem

Let K be algebraically closed, $R = \mathrm{Sym}(R_1)/I$ where I has height c and does not contain linear forms and assume that X is connected in codimension 1. If R is reduced, then the minimal quadratic generators of I are $\leq \binom{c+1}{2}$ and R has multiplicity $\geq c+1$; if equality holds, then R is Cohen-Macaulay.

Serre conditions

For $r \in \mathbb{N}$, we say that R *satisfies the Serre condition* (S_r) if:

$$\operatorname{depth} R_{\mathfrak{p}} \geq \min\{\dim R_{\mathfrak{p}}, r\} \quad \forall\ \mathfrak{p} \in \operatorname{Spec} R.$$

It turns out that this is equivalent to $\operatorname{depth} R \geq \min\{\dim R, r\}$ and

$$\operatorname{depth} \mathcal{O}_{X,x} \geq \min\{\dim \mathcal{O}_{X,x}, r\} \quad \forall\ x \in X.$$

In particular, if X is nonsingular, R satisfies the Serre condition (S_r) if and only if $\operatorname{depth} R \geq \min\{\dim R, r\}$.

Serre conditions

Notice that R is Cohen-Macaulay if and only if R satisfies condition (S_i) for all $i \in \mathbb{N}$. Since if R is CM $h_i \geq 0$ for all $i \in \mathbb{N}$, it is natural to ask:

Question

If R satisfies (S_r), is it true that $h_i \geq 0$ for all $i = 0, \ldots, r$?

The above question is known to have a positive answer for Stanley-Reisner rings R (i.e. if R is defined by squarefree monomial ideals) by a result of Murai and Terai ...

It is too optimistic to expect a positive answer to the previous question in general though: Let $S = K[x_i, y_i : i = 1, \ldots, r+1]$ and $I \subseteq S$ the ideal

$$I = (x_1, \ldots, x_{r+1})^2 + (x_1 y_1 + \ldots + x_{r+1} y_{r+1}).$$

One can check that $R = S/I$ has dimension $(r+1)$, satisfies (S_r) and has h-vector

$$(1, r+1, -1).$$

Moreover, such an R has Castelnuovo-Mumford regularity $\operatorname{reg} R = 1$ and is Buchsbaum. So the question must be adjusted:

Question

If R satisfies (S_r) and X has nice singularities, is it true that $h_i \geq 0$ for all $i = 0, \ldots, r$?

The main result

Theorem (Dao-Ma-_)

Let R satisfy Serre condition (S_r). Suppose either

- K has characteristic 0 and X is Du Bois, or
- K has positive characteristic and X is globally F-split.

Then $h_i \geq 0$ for all $i = 0, \ldots, r$ and the degree of $X \subset \mathbb{P}^n$ is at least $h_0 + h_1 + \ldots + h_{r-1}$. Furthermore, if $\operatorname{reg} R < r$, or if $h_i = 0$ for some $1 \leq i \leq r$, then R is Cohen-Macaulay.

The characteristic 0 version of the previous result is much stronger than the positive characteristic one: in fact, we have the following:

- If $\mathrm{char}(K) = 0$, R Stanley-Reisner ring $\Rightarrow$ X Du Bois.
- If $\mathrm{char}(K) = 0$, X nonsingular $\Rightarrow$ X Du Bois.
- If $\mathrm{char}(K) > 0$, R S-R ring $\Rightarrow$ X globally F-split.
- If $\mathrm{char}(K) > 0$, X nonsingular $\not\Rightarrow$ X globally F-split.

The definition of Du Bois variety (over a field of characteristic 0) is quite involved, it is worth, however, to notice the following:

We say that X is *locally Stanley-Reisner* if $\widehat{\mathcal{O}_{X,x}} \cong K[[\Delta_x]]$ for all $x \in X$ (for some simplicial complex Δ_x). We have:

- X locally Stanley-Reisner $\Rightarrow$ X Du Bois.
- R Stanley-Reisner ring $\Rightarrow$ X locally Stanley-Reisner.
- X nonsingular $\Rightarrow$ X locally Stanley-Reisner.

The condition MT_r

Let M be a finitely generated graded S-module, where $S = K[X_0, \dots, X_n]$. We say that M satisfies the condition MT_r if

$$\operatorname{reg} \operatorname{Ext}_S^{n+1-i}(M, \omega_S) \leq i - r \quad \forall\ i = 0, \dots, \dim M - 1.$$

This notion is good for several reasons:

- The condition MT_r does not depend on S.
- The condition MT_r is preserved by taking general hyperplane sections.
- The condition MT_r is preserved by saturating.

The condition MT_r

Lemma (Murai-Terai, Dao-Ma-_)

Let M be a finitely generated graded S-module generated in degree ≥ 0 with h-vector $(h_0, \ldots, h_s)$ satisfying MT_r. Then

- $h_i \geq 0$ for all $i \leq r$.
- $h_r + h_{r+1} + \ldots + h_s \geq 0$, or equivalently the multiplicity of M is at least $h_0 + h_1 + \ldots + h_{r-1}$.

Furthermore, if reg $M < r$ or M is generated in degree 0 and $h_i = 0$ for some $i \leq r$, then M is Cohen-Macaulay.

The condition MT_r

Let $\dim_K R_1 = n+1$ and $S = K[X_0, \ldots, X_n]$. We want to show that, if R satisfies Serre condition (S_r), then it also satisfies MT_r, namely

$$\operatorname{reg} \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \leq i - r \quad \forall\ i = 0, \ldots, \dim R - 1,$$

provided X is Du Bois (in characteristic 0) or globally F-split (in positive characteristic). By the previous lemma this would imply the desired result.

Dimensions of Ext's

It is simple to show that

$$\dim \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \leq i \quad \forall\ i = 0, \ldots, \dim R - 1,$$

with equality holding iff $\dim R/\mathfrak{p} = i$ for some associated prime $\mathfrak{p}$ of R. A similar argument, plus the fact that R is unmixed as soon as it satisfies (S_2), shows that the following are equivalent for any natural number $r \geq 2$:

1. R satisfies Serre condition (S_r).
2. $\dim \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \leq i - r \quad \forall\ i \in \mathbb{N}$.

Regularity of Ext's

So, under our assumptions on X, in order to prove that R satisfies MT_r provided it satisfies (S_r), it is enough to show that

$$\operatorname{reg} \operatorname{Ext}_S^{n+1-i}(R,\omega_S) \leq \dim \operatorname{Ext}_S^{n+1-i}(R,\omega_S) \quad \forall\, i = 0,\ldots,\dim R - 1.$$

We show more:

Dao-Ma-_

Let $\mathfrak{m}$ be the homogeneous maximal ideal of S and assume either

- K has characteristic 0 and X is Du Bois, or
- K has positive characteristic and X is globally F-split.

Then $H^j_{\mathfrak{m}}(\operatorname{Ext}^i_S(R,\omega_S))_{>0} = 0$ for all $i,j \in \mathbb{N}$. In particular, $\operatorname{reg} \operatorname{Ext}^i_S(R,\omega_S) \leq \dim \operatorname{Ext}^i_S(R,\omega_S)$ for all $i \in \mathbb{N}$.

A corollary

Corollary

Let $R = S/I$ satisfies (S_r) and assume I has height c and does not contain elements of degree $< r$. Suppose either

- K has characteristic 0 and X is Du Bois, or
- K has positive characteristic and X is globally F-split.

Then, the minimal generators of I of degree r are $\leq \binom{c+r-1}{r}$ and R has multiplicity $\geq \sum_{i=0}^{r-1} \binom{c+i-1}{i}$; if equality holds, then R is Cohen-Macaulay.

If $r = 2$, the above corollary is true just assuming that X is geometically reduced...

Questions:

Is it true that $h_i \geq 0$ for all $i \leq r$ provided R satisfies (S_r) and either

1. K has characteristic 0 and X is Du Bois in codimension $r-2$, or
2. K has positive characteristic and X has F-injective singularities ???

THANK YOU !

Some references:

1. H. Dao, L. Ma, M. Varbaro *Regularity, singularities and h-vector of graded algebras*, arXiv:1901.01116.
2. S. Murai, N. Terai, *h-vectors of simplicial complexes with Serre's conditions*, Math. Res. Lett. 16 (2009), no. 6, 1015-1028.