Gröbner deformations

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Notation and basic definitions

- $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- K any field.
- $R = K[X_1, \ldots, X_n]$ the polynomial ring in n variables over K.
- A *monomial* of R is an element $X^u := X_1^{u_1} \cdots X_n^{u_n} \in R$, where $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$.
- $\mathrm{Mon}(R)$ is the set of monomials of R.
- A *term* of R is an element of the form $a\mu \in R$ where $a \in K$ and μ is a monomial.

Notice that every $f \in R$ can be written as a sum of terms: there exists a unique (finite) subset $\mathrm{supp}(f) \subset \mathrm{Mon}(R)$ such that:

$$f = \sum_{\mu \in \mathrm{supp}(f)} a_\mu \mu, \qquad a_\mu \in K \setminus \{0\}.$$

In the above representation, the only lack of uniqueness is the order of the terms.

Definition

A *monomial order* on R is a total order $<$ on $\mathrm{Mon}(R)$ such that:

(i) $1 \leq \mu$ for every $\mu \in \mathrm{Mon}(R)$;

(ii) If $\mu_1, \mu_2, \nu \in \mathrm{Mon}(R)$ such that $\mu_1 \leq \mu_2$, then $\mu_1\nu \leq \mu_2\nu$.

Notice that, if $<$ is a monomial order on R and μ, ν are monomials such that $\mu|\nu$, then $\mu \leq \nu$: indeed $1 \leq \nu/\mu$, so

$$\mu = 1 \cdot \mu \leq (\nu/\mu) \cdot \mu = \nu.$$

Notation and basic definitions

Typical examples of monomial orders are the following: given monomials $\mu = X_1^{u_1} \cdots X_n^{u_n}$ and $\nu = X_1^{v_1} \cdots X_n^{v_n}$ we define:

- The *lexicographic order* (Lex) by $\mu <_{\text{Lex}} \nu$ iff $u_k < v_k$ for some k and $u_i = v_i$ for any $i < k$.
- The *degree lexicographic order* (DegLex) by $\mu <_{\text{DegLex}} \nu$ iff $\deg(\mu) < \deg(\nu)$ or $\deg(\mu) = \deg(\nu)$ and $\mu <_{Lex} \nu$.
- The *(degree) reverse lexicographic order* (RevLex) by $\mu <_{\text{RevLex}} \nu$ iff $\deg(\mu) < \deg(\nu)$ or $\deg(\mu) = \deg(\nu)$ and $u_k > v_k$ for some k and $u_i = v_i$ for any $i > k$.

Example

In $K[X, Y, Z]$, assuming $X > Y > Z$, we have $X^2 >_{\text{Lex}} XZ >_{\text{Lex}} Y^2$, while $X^2 >_{\text{RevLex}} Y^2 >_{\text{RevLex}} XZ$.

Proposition

A monomial order on R is a well-order on $\mathrm{Mon}(R)$. That is, any nonempty subset of $\mathrm{Mon}(R)$ has a minimum. Equivalently, all descending chains of monomials in R terminate.

Proof: Let $\emptyset \neq N \subset \mathrm{Mon}(R)$, and $I \subset R$ be the ideal generated by N. By Hilbert basis theorem, I is generated by a finite number of monomials of N. Since a monomial order refines divisibility, the minimum of such finitely many monomials is also the minimum of N. $\square$

Notation and basic definitions

From now on, we fix a monomial order $<$ on R, so that every polynomial $0 \neq f \in R$ can be written uniquely as

$$f = a_1\mu_1 + \ldots + a_k\mu_k$$

with $a_i \in K \setminus \{0\}$, $\mu_i \in \text{Mon}(R)$ and $\mu_1 > \mu_2 > \ldots > \mu_k$.

Definition

The *initial monomial* of f is $\text{in}(f) = \mu_1$. Furthermore, its *initial coefficient* is $\text{inic}(f) = a_1$ and its *initial term* is $\text{init}(f) = a_1\mu_1$.

Notice that, for all $f, g \in R$:

- $\text{inic}(f)\,\text{in}(f) = \text{init}(f)$.
- $\text{in}(fg) = \text{in}(f)\,\text{in}(g)$.
- $\text{in}(f+g) \leq \max\{\text{in}(f), \text{in}(g)\}$.

Notation and basic definitions

Example

If $f = X_1 + X_2X_4 + X_3^2$, we have:

- $\text{in}(f) = X_1$ with respect to Lex.
- $\text{in}(f) = X_2X_4$ with respect to DegLex.
- $\text{in}(f) = X_3^2$ with respect to RevLex.

Example

If $f = X^2 + XY + Y^2 \in K[X, Y]$, then we have:

- $\text{in}(f) = X^2$ if $X > Y$.
- $\text{in}(f) = Y^2$ if $Y > X$.

In particular, $XY \neq \text{in}(f)$ for all monomial orders.

Gröbner bases and Buchberger algorithm

Definition

If I is an ideal of R, then the monomial ideal $\text{in}(I) \subset R$ generated by $\{\text{in}(f) : f \in I\}$ is named the *initial ideal* of I.

Definition

Polynomials $f_1, \ldots, f_m$ of an ideal $I \subset R$ are a *Gröbner basis* of I if $\text{in}(I) = (\text{in}(f_1), \ldots, \text{in}(f_m))$.

Example

Consider the ideal $I = (f_1 = X^2 - Y^2, f_2 = XZ - Y^2)$ of $K[X, Y, Z]$. For Lex with $X > Y > Z$ the polynomials f_1, f_2 are not a Gröbner basis of I, indeed $XY^2 = \text{in}(Zf_1 - Xf_2)$ is a monomial of $\text{in}(I)$ which is not in $(\text{in}(f_1) = X^2, \text{in}(f_2) = XZ)$. For RevLex with $X > Y > Z$, it turns out that $\text{in}(I) = (X^2, Y^2)$, so f_1 and f_2 are a Gröbner basis of I in this case.

Gröbner bases and Buchberger algorithm

Remark

The Noetherianity of R implies that any ideal in R has a finite Gröbner basis.

There is a way to compute a Gröbner basis of an ideal I starting from a system of generators of I, namely the *Buchsberger algorithm*; it also checks if such a system of generators is already a Gröbner basis. We will develop the algorithm in the next few slides:

Definition

Let $f_1, \ldots, f_m \in R$. A polynomial $r \in R$ is a *reduction of* $g \in R$ *modulo* $f_1, \ldots, f_m$ if there exist $q_1, \ldots, q_m \in R$ satisfying:

- $g = q_1 f_1 + \ldots + q_m f_m + r$;
- $\text{in}(q_i f_i) \leq \text{in}(g)$ for all $i = 1, \ldots, m$;
- For all $i = 1, \ldots, m$, $\text{in}(f_i)$ does not divide $\mu \; \forall \; \mu \in \text{supp}(r)$.

Lemma

Let $f_1, \ldots, f_m \in R$. Every polynomial $g \in R$ admits a reduction modulo $f_1, \ldots, f_m$.

Proof: Let $J = (\mathrm{in}(f_1), \ldots, \mathrm{in}(f_m))$. We start with $r = g$ and apply the *reduction algorithm*:

(1) If $\mathrm{supp}(r) \cap J = \emptyset$, we are done: r is the desired reduction.

(2) Otherwise choose $\mu \in \mathrm{supp}(r) \cap J$ and let $b \in K$ be the coefficient of μ in the monomial representation of r. Choose i such that $\mathrm{in}(f_i) \mid \mu$ and set $r' = r - a\nu f_i$ where $\nu = \mu / \mathrm{in}(f_i)$ and $a = b / \mathrm{inic}(f_i)$. Then replace r by r' and go to (1).

This algorithm terminates after finitely many steps since it replaces the monomial μ by a linear combination of monomials that are smaller in the monomial order, and all descending chains of monomials in R terminate. □

Example

Once again, we take $R = K[X, Y, Z]$, $f_1 = X^2 - Y^2$ and $f_2 = XZ - Y^2$, and we consider Lex with $X > Y > Z$. Set $g = X^2Z$. Then $g = Zf_1 + Y^2Z$, but $g = Xf_2 + XY^2$ as well. Both these equations yield reductions of g, namely XY^2 and Y^2Z. Thus a polynomial can have several reductions modulo f_1, f_2.

The reduction of $g \in R$ modulo $f_1, \ldots, f_m$ is unique when $f_1, \ldots, f_m$ is a Gröbner basis...

Gröbner bases and Buchberger algorithm

Proposition

Let I be an ideal of R, $f_1, \dots, f_m \in I$ and $J = (\mathrm{in}(f_1), \dots, \mathrm{in}(f_m))$. Then the following are equivalent:

(a) $f_1, \dots, f_m$ form a Gröbner basis of I;

(b) every $g \in I$ reduces to 0 modulo $f_1, \dots, f_m$;

(c) the monomials μ, $\mu \notin J$, are linearly independent modulo I.

If the equivalent conditions (a), (b), (c) hold, then:

(d) Every element of R has a unique reduction modulo $f_1, \dots, f_m$.

(e) The reduction depends only on I and the monomial order.

Proof: Check (a) $\implies$ (c) $\implies$ (b) as an exercise.

(b) $\implies$ (a) Let $g \in I$, $g \neq 0$. If g reduces to 0, then we have

$$g = q_1 f_1 + \cdots + q_m f_m$$

such that $\text{in}(q_i f_i) \leq \text{in}(g)$ for all i. But the monomial $\text{in}(g)$ must appear on the right hand side as well, and this is only possible if $\text{in}(g) = \text{in}(q_i f_i) = \text{in}(q_i)\,\text{in}(f_i)$ for at least one i. In other words, $\text{in}(g)$ must be divisible by $\text{in}(f_i)$ for some i. Hence $\text{in}(I) = J$.

Check (c) $\implies$ (d), (e) as an exercise. $\square$

Corollary

If $f_1, \ldots, f_m$ is a Gröbner basis of an ideal $I \subset R$ then $I = (f_1, \ldots, f_m)$.

Corollary

Let $I \subset R$ be an ideal and $<_1, <_2$ monomial orders of R. If $\text{in}_{<_1}(I) \subset \text{in}_{<_2}(I)$, then $\text{in}_{<_1}(I) = \text{in}_{<_2}(I)$.

Proof: By the previous proposition, the sets A_i of monomials of R not in $\text{in}_{<_i}(I)$ are K-bases of R/I for each $i = 1, 2$. Since $A_1 \supset A_2$, we must have $A_1 = A_2$. $\square$

Corollary

Let $I_1, I_2 \subset R$ be ideals and $<$ a monomial order of R. If $I_1 \subset I_2$ and $\text{in}_<(I_1) = \text{in}_<(I_2)$, then $I_1 = I_2$.

Proof: By the previous proposition, the set A of monomials of R not in $\text{in}_<(I_1) = \text{in}_<(I_2)$ are K-bases of R/I_i for each $i = 1, 2$. Since $I_1 \subset I_2$, we must have $I_1 = I_2$. $\square$

Gröbner bases and Buchberger algorithm

Definition

The *S-polynomial* of two elements $f, g \in R$ is defined as

$$S(f,g) = \frac{\operatorname{lcm}(\operatorname{in}(f), \operatorname{in}(g))}{\operatorname{init}(f)} f - \frac{\operatorname{lcm}(\operatorname{in}(f), \operatorname{in}(g))}{\operatorname{init}(g)} g$$

Proposition

Let $f_1, \ldots, f_m \in R$ and $I = (f_1, \ldots, f_m)$. Then the following are equivalent:

(a) $f_1, \ldots, f_m$ form a Gröbner basis of I.

(b) For all $1 \leq i < j \leq m$, $S(f_i, f_j)$ reduces to 0 modulo $f_1, \ldots, f_m$.

Proof: (a) $\implies$ (b): It follows since $S(f_i, f_j) \in I$.

(b) $\implies$ (a): We need to show that every $g \in I$ reduces to 0 modulo the f_k's. Since $g \in I$, we have $g = a_1 f_1 + \ldots + a_m f_m$ for some $a_k \in R$. Among such representations, we can choose one minimizing $\mu := \max\{\mathrm{in}(a_i f_i) : i = 1, \ldots, m\}$ and, among these, minimizing $s := |\{i = 1, \ldots, m | \, \mathrm{in}(a_i f_i) = \mu\}|$. By contradiction, suppose $\mu > \mathrm{in}(g)$. In this case $s \geq 2$, so there exist $i < j$ such that $\mathrm{in}(a_i f_i) = \mathrm{in}(a_j f_j) = \mu$. Set $c := \mathrm{inic}(a_i f_i)$ and notice that $\mu = \nu \cdot \mathrm{lcm}(\mathrm{in}(f_i), \mathrm{in}(f_j))$ for some $\nu \in \mathrm{Mon}(R)$. Let

$$S(f_i, f_j) = q_1 f_1 + \ldots + q_m f_m$$

the reduction of $S(f_i, f_j)$ (so that $\mathrm{in}(q_k f_k) \leq \mathrm{in}(S(f_i, f_j))$ which is less than $\alpha_{ij} := \mathrm{lcm}(\mathrm{in}(f_i), \mathrm{in}(f_j))$ for all k). From this we get a representation $g = a'_1 f_1 + \ldots + a'_m f_m$ contradicting the minimality of μ and s where $a'_i = a_i - \frac{c\nu\alpha_{ij}}{\mathrm{init}(f_i)} + c\nu q_i$, $a'_j = a_j + \frac{c\nu\alpha_{ij}}{\mathrm{init}(f_j)} + c\nu q_j$ and $a'_k = a_k + c\nu q_k$ for $i \neq k \neq j$. $\square$

Initial ideals with respect to weights

Fix $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ a *weight vector*. If $\mu = X^u \in \mathrm{Mon}(R)$ with $u = (u_1, \dots, u_n)$ then we set $w(\mu) := w_1 u_1 + \ldots + w_n u_n$. If $0 \neq f \in R$ we set $w(f) := \max\{w(\mu) : \mu \in \mathrm{supp}(f)\}$ and

$$\mathrm{init}_w(f) = \sum_{\substack{\mu \in \mathrm{supp}(f) \\ w(\mu) = w(f)}} a_\mu \mu,$$

where $f = \sum_{\mu \in \mathrm{supp}(f)} a_\mu \mu$.

Example

If $w = (2, 1)$ and $f = X^3 + 2X^2Y^2 - Y^5 \in \mathbb{Q}[X, Y]$ then $\mathrm{init}_w(f) = X^3 + 2X^2Y^2$.

Given an ideal $I \subset R$ we set $\mathrm{in}_w(I) = (\mathrm{init}_w(f) : f \in I) \subset R$.

As we will see, the passage from an ideal I to $\mathrm{in}_w(I)$ can be seen as a "continuous" degenerative process. Before explaining it, we will show that, given a monomial order $<$ on R and an ideal $I \subset R$, we can always find a suitable $w \in (\mathbb{N}_{>0})^n$ such that $\mathrm{in}_w(I) = \mathrm{in}_<(I)$.

Example

Let us find a weight vector that picks the largest monomial in every subset of monomials of degree $\leq d$ in $K[X, Y, Z]$ for the lexicographic order determined by $X > Y > Z$. We give weight 1 to Z. Since $Y > Z^d$, we give weight $d + 1$ to Y. Since $X > Y^d$ and $w(Y^d) = d(d + 1)$, we must choose $w(X) = d(d + 1) + 1$. It is not hard to check that $w = (d(d + 1) + 1, d + 1, 1)$ indeed solves our problem.

Initial ideals with respect to weights

Given $w \in \mathbb{N}^n$ and $<$ a monomial order, we define another monomial order on R as

$$\mu <_w \nu \iff \begin{cases} w(\mu) < w(\nu) \\ w(\mu) = w(\nu) \text{ and } \mu < \nu \end{cases} .$$

Lemma

For an ideal $I \subset R$, if $\mathrm{in}_w(I) \subset \mathrm{in}_<(I)$ or $\mathrm{in}_w(I) \supset \mathrm{in}_<(I)$, then $\mathrm{in}_w(I) = \mathrm{in}_<(I)$.

Proof: By applying $\mathrm{in}_<(-)$ on both sides we get, for example, $\mathrm{in}_{<_w}(I) = \mathrm{in}_<(\mathrm{in}_w(I)) \supset \mathrm{in}_<(\mathrm{in}_<(I)) = \mathrm{in}_<(I)$. So the equality $\mathrm{in}_<(\mathrm{in}_w(I)) = \mathrm{in}_<(\mathrm{in}_<(I))$ must hold, and because $\mathrm{in}_w(I) \supset \mathrm{in}_<(I)$ we must have $\mathrm{in}_w(I) = \mathrm{in}_<(I)$. $\square$

Initial ideals with respect to weights

Lemma

Let $P \subset \mathbb{R}^n$ be the convex hull of some vectors $u^1, \ldots, u^m \in \mathbb{N}^n$. Then $X^u \leq \max\{X^{u^1}, \ldots, X^{u^m}\}$ for any $u \in P \cap \mathbb{N}^n$.

Proof: If $u \in P \cap \mathbb{N}^n$, then $u = \sum_{i=1}^m \lambda_i u^i$ with $\lambda_i \in \mathbb{Q}_{\geq 0}$ and $\sum_{i=1}^m \lambda_i = 1$. If $\lambda_i = a_i/b_i$ with $a_i \in \mathbb{N}$, $b_i \in \mathbb{N} \setminus \{0\}$, then we have

$$bu = \sum_{i=1}^m a_i' u^i,$$

where $b = b_1 \cdots b_m$ and $a_i' = a_i(b/b_i)$. If, by contradiction, $X^u > X^{u^i}$ for all $i = 1, \ldots, m$, then

$$(X^u)^b > (X^{u^1})^{a_1'} \cdots (X^{u^m})^{a_m'}$$

(because $b = \sum_{i=1}^m a_i'$) but this contradicts the fact that these two monomials are the same. $\square$

Initial ideals with respect to weights

Proposition

Given a monomial order $>$ on R and $\mu_i, \nu_i \in \mathrm{Mon}(R)$ such that $\mu_i > \nu_i$ for $i = 1, \ldots, k$, there exists $w \in (\mathbb{N}_{>0})^n$ such that $w(\mu_i) > w(\nu_i)\ \forall\ i = 1, \ldots, k$. Consequently, given an ideal $I \subset R$ there exists $w \in (\mathbb{N}_{>0})^n$ such that $\mathrm{in}_<(I) = \mathrm{in}_w(I)$.

Proof: Notice that $\mu_i > \nu_i \iff \prod_j \mu_j > \nu_i \prod_{j \neq i} \mu_j$ and $w(\mu_i) > w(\nu_i) \iff w(\prod_j \mu_j) > w(\nu_i \prod_{j \neq i} \mu_j)$, so we can assume that μ_i is the same monomial μ for all $i = 1, \ldots, k$. If $\mu = X^u$ and $\nu_i = X^{v^i}$, consider $C = u + (\mathbb{R}_{\geq 0})^n \subset \mathbb{R}^n$ and $P \subset \mathbb{R}^n$ the convex hull of u and $v^1, \ldots, v^k$. We claim that $C \cap P = \{u\}$. Suppose that $v \in C \cap P$. We can assume that $v \in \mathbb{Q}^n$, so that there is $N \in \mathbb{N}$ big enough such that $Nv \in \mathbb{N}$. Let $\nu = X^{Nv}$. Since $v \in C$, ν is divided by $\mu^N = X^{Nu}$, so $\nu \geq \mu^N$. On the other hand, $v \in P \implies Nv \in NP$, so $\nu \leq \max\{Nu, Nv^i : i = 1, \ldots, k\} = Nu$ by the previous lemma, so $\nu = \mu^N$, that is $v = u$.

Therefore there is a hyperplane passing through u separating C and P, that is there is $w \in (\mathbb{R}^n)^*$ such that

$$w(v) > w(u) > w(v^i)$$

for all $v \in C \setminus \{u\}$ and $i = 1, \ldots, k$. Of course we can pick $w = (w_1, \ldots, w_n) \in \mathbb{Q}^n$; furthermore the first inequalities yield $w_i > 0$ for all $i = 1, \ldots, n$. After taking a suitable multiple, so, we can assume $w \in (\mathbb{N}_{>0})^n$ is our desired weight vector.

For the last part of the statement, let $f_1, \ldots, f_m$ be a Gröbner basis of I. By the first part, there is $w \in (\mathbb{N}_{>0})^n$ such that $w(\mu) > w(\nu)$ where $\mu = \mathrm{in}(f_i)$ and $\nu \in \mathrm{supp}(f_i) \setminus \{\mu\}$ for all $i = 1, \ldots, m$. So $\mathrm{in}_<(I) \subset \mathrm{in}_w(I)$, hence $\mathrm{in}_<(I) = \mathrm{in}_w(I)$. $\square$

Initial ideals with respect to weights

Let us extend R to $P = R[t]$ by introducing a *homogenizing variable* t. The *w-homogenization* of $f = \sum_{\mu \in \mathrm{supp}(f)} a_\mu \mu \in R$ is

$$\mathrm{hom}_w(f) = \sum_{\mu \in \mathrm{supp}(f)} a_\mu \mu t^{w(f)-w(\mu)} \in P.$$

Example

Let $f = X^2 - XY + Z^2 \in K[X, Y, Z]$. We have:

- $\mathrm{hom}_w(f) = X^2 - XY + Z^2 t^2$ if $w = (2, 2, 1)$.
- $\mathrm{hom}_w(f) = X^2 - XYt^2 + Z^2 t^6$ if $w = (4, 2, 1)$.

Given an ideal $I \subset R$, $\mathrm{hom}_w(I) \subset P$ denotes the ideal generated by $\mathrm{hom}_w(f)$ with $f \in I$. For its study, we extend the weight vector w to w' on P by $w'(t) = 1$, so that $\mathrm{hom}_w(I)$ is a w'-homogeneous ideal of P, where the grading is $\deg(X_i) = w_i$ and $\deg(t) = 1$.

Because $P/\hom_w(I)$ is a w'-graded P-module, it is also a graded $K[t]$-module (w.r.t. the standard grading on $K[t]$). So $t-a$ is not a zero-divisor on $P/\hom_w(I)$ for any $a \in K \setminus \{0\}$. We want to show that also t is not a zero-divisor on $P/\hom_w(I)$ as well, and in order to do so it is useful to consider the *dehomogenization map*:

$$\pi : P \longrightarrow R$$
$$F(X_1, \dots, X_n, t) \mapsto F(X_1, \dots, X_n, 1).$$

Remark

1. $\pi(\hom_w(f)) = f \ \forall \ f \in R$. So, $\pi(\hom_w(I)) = I$.
2. If $F \in P \setminus tP$ is w'-homogeneous, then $\hom_w(\pi(F)) = F$; moreover, if $r \in \mathbb{N}$ and $G = t^r F$, $\hom_w(\pi(G))t^r = G$.

Summarizing, for $F \in P$ we have $F \in \hom_w(I) \iff \pi(F) \in I$.

Initial ideals with respect to weights

Proposition

Given an ideal I of R, the element $t - a \in K[t]$ is not a zero divisor on $P/\operatorname{hom}_w(I)$ for every $a \in K$. Furthermore:

- $P/\big(\operatorname{hom}_w(I) + (t)\big) \cong R/\operatorname{in}_w(I)$.
- $P/\big(\operatorname{hom}_w(I) + (t - a)\big) \cong R/I$ for all $a \in K \setminus \{0\}$.

Proof: For the first assertion, we need to show it just for $a = 0$: Let $F \in P$ such that $tF \in \operatorname{hom}_w(I)$. Then $\pi(tF) \in I$, so, since $\pi(F) = \pi(tF)$, $F \in \operatorname{hom}_w(I)$.

For $P/(\operatorname{hom}_w(I) + (t)) \cong R/\operatorname{in}_w(I)$ it is enough to check that $\operatorname{hom}_w(I) + (t) = \operatorname{in}_w(I) + (t)$. This is easily seen since for every $f \in R$ the difference $\operatorname{hom}_w(f) - \operatorname{init}_w(f)$ is divisible by t.

To prove that $P/(\hom_w(I) + (t-a)) \cong R/I$ for every $a \in K \setminus \{0\}$, we consider the graded isomorphism $\psi : R \to R$ induced by $\psi(X_i) = a^{-w_i}X_i$. Of course $\psi(\mu) = a^{-w(\mu)}\mu \ \forall\ \mu \in \mathrm{Mon}(R)$ and $\hom_w(f) - a^{w(f)}\psi(f)$ is divisible by $t-a$ for all $f \in R$. So $\hom_w(I) + (t-a) = \psi(I) + (t-a)$, which implies the desired isomorphism. $\square$

Remark

Since a module over a PID is flat iff it has no torsion, the proposition above says that $P/\hom_w(I)$ is a flat $K[t]$-module, and that it defines a flat family over $K[t]$ with generic fiber R/I and special fiber $R/\operatorname{in}_w(I)$.

Next we want to show that local cohomology cannot shrink passing to the initial ideal. We need the following first:

Lemma

Let A be a ring, M, N A-modules and $a \in \mathrm{ann}(N) \subset A$ a non-zero-divisor on M as well as on A. Then, for all $i \geq 0$,

$$\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^i_{A/aA}(M/aM, N).$$

Proof: Let $F_\bullet$ be a free resolution of M. The Ext modules on the left hand side are the cohomology modules of $\mathrm{Hom}_A(F_\bullet, N)$, which is a complex of A-modules isomorphic to $\mathrm{Hom}_{A/aA}(F_\bullet/aF_\bullet, N)$ because a annihilates N. However $F_\bullet/aF_\bullet$ is a free resolution of the A/aA-module M/aM since a is a non-zero-divisor on M as well as on A, so the cohomology modules of the latter complex are the Ext modules on the right hand side. □

Initial ideals with respect to weights

Let us give a graded structure to $R = K[X_1, \dots, X_n]$ by putting $\deg(X_i) = g_i$ where $g = (g_1, \dots, g_n)$ is a vector of positive integers (so that $\mathfrak{m} = (X_1, \dots, X_n)$ is the unique homogeneous maximal ideal of R). If $I \subset R$ is a g-homogeneous ideal, then $\hom_w(I) \subset P$ is homogeneous with respect to the *bi-graded* structure on P given by $\deg(X_i) = (g_i, w_i)$ and $\deg(t) = (0, 1)$. So $S = P/\hom_w(I)$ and $\mathrm{Ext}^i_P(S, P)$ are finetely generated bi-graded P-modules.

Notice that, given a finitely generated bi-graded P-module M, $M_{(j,*)} = \bigoplus_{k \in \mathbb{Z}} M_{(j,k)}$ is a finitely generated graded (w.r.t. the standard grading) $K[t]$-module for all $j \in \mathbb{Z}$. Finally, if N is a finitely generated $K[t]$-module, $N \cong K[t]^a \oplus T$ for $a \in \mathbb{N}$ and some finitely generated torsion $K[t]$-module T (since $K[t]$ is a PID). If N is also graded, then $T \cong \bigoplus_{k \in \mathbb{N}_{>0}} (K[t]/(t^k))^{b_k}$.

Initial ideals with respect to weights

From now, let us fix a g-homogeneous ideal $I \subset R$ and denote $P/\hom_w(I)$ by S. From the above discussion, for all $i,j \in \mathbb{Z}$:

$$\operatorname{Ext}^i_P(S,P)_{(j,*)} \cong K[t]^{a_{i,j}} \oplus \left(\bigoplus_{k \in \mathbb{N}_{>0}} (K[t]/(t^k))^{b_{i,j,k}} \right)$$

for some natural numbers $a_{i,j}$ and $b_{i,j,k}$. Let $b_{i,j} = \sum_{k \in \mathbb{N}_{>0}} b_{i,j,k}$.

Theorem

With the above notation, for any $i,j \in \mathbb{Z}$ we have:

- $\dim_K(\operatorname{Ext}^i_R(R/I,R)_j) = a_{i,j}$.
- $\dim_K(\operatorname{Ext}^i_R(R/\operatorname{in}_w(I),R)_j) = a_{i,j} + b_{i,j} + b_{i+1,j}$.

In particular, $\dim_K(\operatorname{Ext}^i_R(R/I,R)_j) \leq \dim_K(\operatorname{Ext}^i_R(R/\operatorname{in}_w(I),R)_j)$ and $\dim_K(H^i_{\mathfrak{m}}(R/I)_j) \leq \dim_K(H^i_{\mathfrak{m}}(R/\operatorname{in}_w(I))_j)$.

Initial ideals with respect to weights

Proof: Letting x be t or $t-1$ we have the short exact sequence

$$0 \to P \xrightarrow{\cdot x} P \to P/xP \to 0.$$

The long exact sequence of $\mathrm{Ext}_P(S,-)$ associated to it, gives us the following short exact sequences for all $i \in \mathbb{Z}$:

$$0 \to \mathrm{Coker}\,\alpha_{i,x} \to \mathrm{Ext}^i_P(S, P/xP) \to \mathrm{Ker}\,\alpha_{i+1,x} \to 0,$$

where $\alpha_{k,x}$ is the multiplication by x on $\mathrm{Ext}^k_P(S,P)$. We can restrict the above exact sequences to the degree $(j,*)$ for any $j \in \mathbb{Z}$ getting:

$$0 \to (\mathrm{Coker}\,\alpha_{i,x})_{(j,*)} \to (\mathrm{Ext}^i_P(S, P/xP))_{(j,*)} \to (\mathrm{Ker}\,\alpha_{i+1,x})_{(j,*)} \to 0.$$

Notice that we have:

- $(\mathrm{Coker}\,\alpha_{i,t})_{(j,*)} \cong K^{a_{i,j}+b_{i,j}}$ and $(\mathrm{Ker}\,\alpha_{i+1,t})_{(j,*)} \cong K^{b_{i+1,j}}$.
- $(\mathrm{Coker}\,\alpha_{i,t-1})_{(j,*)} \cong K^{a_{i,j}}$ and $(\mathrm{Ker}\,\alpha_{i+1,t-1})_{(j,*)} = 0$.

Initial ideals with respect to weights

Therefore, for all $i, j \in \mathbb{Z}$, we got:

- $(\mathrm{Ext}^i_P(S, P/tP))_{(j,*)} \cong K^{a_{i,j}+b_{i,j}+b_{i+1,j}}$.
- $(\mathrm{Ext}^i_P(S, P/(t-1)P))_{(j,*)} \cong K^{a_{i,j}}$.

By a previous proposition both t and $t-1$ are non-zero-divisors on S as well on P, hence a previous lemma together with the same proposition imply:

- $(\mathrm{Ext}^i_P(S, P/tP))_{(j,*)} \cong (\mathrm{Ext}^i_{P/tP}(S/tS, P/tP))_{(j,*)}$, which is isomorphic to $(\mathrm{Ext}^i_R(R/\operatorname{in}_w(I), R))_j$.
- $(\mathrm{Ext}^i_P(S, P/(t-1)P))_{(j,*)} \cong (\mathrm{Ext}^i_{P/(t-1)P}(S/(t-1)S, P/(t-1)P))_{(j,*)}$, which is isomorphic to $(\mathrm{Ext}^i_R(R/I, R))_j$.

The thesis follows from this. For the local cohomology statement just observe that by Grothendieck graded duality $H^i_{\mathfrak{m}}(R/J)_j$ is dual as K-vector space to $\mathrm{Ext}^{n-i}_R(R/J, R)_{-|g|-j}$ for any g-homogeneous ideal $J \subset R$ and $i, j \in \mathbb{Z}$ (where $|g| = g_1 + \ldots + g_n$). $\square$

Corollary

If I is a homogeneous ideal of R, then for all $i, j \in \mathbb{Z}$

$$\dim_K(H^i_{\mathfrak{m}}(R/I)_j) \leq \dim_K(H^i_{\mathfrak{m}}(R/\operatorname{in}(I))_j).$$

Next we want to show that, if $\operatorname{in}(I)$ is squarefree, then we have equalities above. In order to do this, we will show that, if $\operatorname{in}_w(I)$ is a squarefree monomial ideal, then $\operatorname{Ext}^i_P(S, P)$ is a flat $K[t]$-module for all $i \in \mathbb{Z}$ (so that the numbers $b_{i,j}$ in the previous theorem would be 0 for all $i, j \in \mathbb{Z}$). Let us recall that a module is flat over a PID (such as $K[t]$) if and only if it has no torsion...

Fiber-full modules and flatness

In the following slides, A is a Noetherian flat $K[t]$-algebra and M a finitely generated A-module which is flat over $K[t]$, and both A and M are graded $K[t]$-modules (think at A and M like they were, with the previous notation, P and S).

Lemma

The following are equivalent:

1. $\operatorname{Ext}^i_A(M, A)$ is a flat over $K[t]$ for all $i \in \mathbb{N}$.
2. $\operatorname{Ext}^i_{A/t^mA}(M/t^mM, A/t^mA)$ is a flat over $K[t]/(t^m)$ $\forall\, i, m \in \mathbb{N}$.

Proof: (1) $\implies$ (2): Since A is flat over $K[t]$, there is a short exact sequence $0 \to A \xrightarrow{\cdot t^m} A \to A/t^mA \to 0$. Consider the induced long exact sequence of $\operatorname{Ext}_A(M, -)$:

$$\cdots \to \operatorname{Ext}^i_A(M, A) \xrightarrow{\cdot t^m} \operatorname{Ext}^i_A(M, A) \to \operatorname{Ext}^i_A(M, A/t^mA)$$
$$\to \operatorname{Ext}^{i+1}_A(M, A) \xrightarrow{\cdot t^m} \operatorname{Ext}^{i+1}_A(M, A) \to \ldots$$

Fiber-full modules and flatness

By (1), $\operatorname{Ext}^k_A(M, A)$ does not have t-torsion for all $k \in \mathbb{N}$, so for all $i \in \mathbb{N}$ we have a short exact sequence

$$0 \to \operatorname{Ext}^i_A(M, A) \xrightarrow{\cdot t^m} \operatorname{Ext}^i_A(M, A) \to \operatorname{Ext}^i_A(M, A/t^m A) \to 0,$$

from which $\operatorname{Ext}^i_A(M, A/t^m A) \cong \dfrac{\operatorname{Ext}^i_A(M, A)}{t^m \operatorname{Ext}^i_A(M, A)}$. It is straightforward to check that the latter is flat over $K[t]/(t^m)$ because (1). Finally, a previous lemma implies that

$$\operatorname{Ext}^i_A(M, A/t^m A) \cong \operatorname{Ext}^i_{A/t^m A}(M/t^m M, A/t^m A).$$

(2) $\implies$ (1): By contradiction, suppose $\operatorname{Ext}^i_A(M, A)$ is not flat over $K[t]$. Because $K[t]$ is a PID, then $\operatorname{Ext}^i_A(M, A)$ has nontrivial torsion. So, by the graded structure of $\operatorname{Ext}^i_A(M, A)$, there exists a nontrivial class $[\phi] \in \operatorname{Ext}^i_A(M, A)$ and $k \in \mathbb{N}$ such that $t^k[\phi] = 0$.

Fiber-full modules and flatness

Let us take a A-free resolution $F_\bullet$ of M, and let $(G^\bullet, \partial^\bullet)$ be the complex $\mathrm{Hom}_A(F_\bullet, A)$, so that $\mathrm{Ext}^i_A(M, A)$ is the ith cohomology module of $G^\bullet$. Then $\phi \in \mathrm{Ker}(\partial^i) \setminus \mathrm{Im}(\partial^{i-1})$ and $t^k\phi \in \mathrm{Im}(\partial^{i-1})$.

Since M and A are flat over $k[t]$, $F_\bullet/t^m F_\bullet$ is a $A/t^m A$-free resolution of $M/t^m M$. Let $(\overline{G^\bullet}, \overline{\partial^\bullet})$ denote the complex $\mathrm{Hom}_{A/t^m A}(F_\bullet/t^m F_\bullet, A/t^m A)$, so that $\mathrm{Ext}^i_{A/t^m A}(M/t^m M, A/t^m A)$ is the ith cohomology module of $\overline{G^\bullet}$, and $\pi^\bullet$ the natural map of complexes from $G^\bullet$ to $\overline{G^\bullet}$. Of course $\pi^i(\phi) \in \mathrm{Ker}(\overline{\partial^i})$ and $t^k\pi^i(\phi) \in \mathrm{Im}(\overline{\partial^{i-1}})$. Now, it is enough to find a positive integer m such that $\pi^i(\phi)$ does neither belong to $\mathrm{Im}(\overline{\partial^{i-1}})$ nor to $t^{m-k}\,\mathrm{Ker}(\overline{\partial^i})$. Indeed, in this case $x = [\pi^i(\phi)]$ would be an element of $\mathrm{Ext}^i_{P/t^m P}(S/t^m S, P/t^m P) \setminus t^{m-k}\,\mathrm{Ext}^i_{P/t^m P}(S/t^m S, P/t^m P)$ such that $t^k x = 0$, and this would contradict the flatness of $\mathrm{Ext}^i_{P/t^m P}(S/t^m S, P/t^m P)$ over $K[t]/(t^m)$.

If $\pi^i(\phi) \in \operatorname{Im}(\overline{\partial^{i-1}})$, then

$$\phi \in \operatorname{Im}(\partial^{i-1}) + t^m G^i = \operatorname{Im}(\partial^{i-1}) + t^m \operatorname{Ker}(\partial^i).$$

Since ϕ does not belong to $\operatorname{Im}(\partial^{i-1})$, Krull's intersection theorem tells us that $\pi^i(\phi)$ cannot belong to $\operatorname{Im}(\overline{\partial^{i-1}})$ for all $m \gg 0$. Analogously, if $\pi^i(\phi) \in t^{m-k} \operatorname{Ker}(\overline{\partial^i})$, then

$$\phi \in t^{m-k} \operatorname{Ker}(\partial^i) + t^m G^i = t^{m-k} \operatorname{Ker}(\partial^i).$$

But $\phi \neq 0$, so, again using Krull's intersection theorem, $\pi^i(\phi) \notin t^{m-k}\overline{G^i}$ for all $m \gg 0$. $\square$

Fiber-full modules and flatness

In the above situation, we say that M is a *fiber-full* A-module if, for any $m \in \mathbb{N}_{>0}$, the natural projection $M/t^m M \to M/tM$ induces injective maps $\mathrm{Ext}^i_A(M/tM, A) \to \mathrm{Ext}^i_A(M/t^m M, A)$ for all $i \in \mathbb{Z}$.

Next we will see that, if M is a fiber-full A-module, then $\mathrm{Ext}^i_A(M, A)$ is flat over $K[t]$ for all $i \in \mathbb{Z}$. This circle of ideas are due (in slightly different contexts) to Ma-Quy and Kollar-Kovacs. After this, we will show that S is a fiber-full P-module provided that $\mathrm{in}_w(I)$ is a squarefree monomial ideal, and this will imply that

$$\dim_K(H^i_{\mathfrak{m}}(R/I)_j) = \dim_K(H^i_{\mathfrak{m}}(R/\mathrm{in}(I))_j) \;\; \forall\, i, j \in \mathbb{Z}$$

whenever $I \subset R$ is a homogeneous ideal such that $\mathrm{in}(I) \subset R$ is a squarefree monomial ideal, a result of Conca and myself.

Fiber-full modules and flatness

The previous lemma says that to show that $\operatorname{Ext}^i_A(M, A)$ is flat over $K[t]$ for all $i \in \mathbb{Z}$ it is enough to show that the $K[t]/(t^m)$-module $\operatorname{Ext}^i_{A/t^m A}(M/t^m M, A/t^m A)$ is flat for all $i \in \mathbb{Z}$ and $m \in \mathbb{N}_{>0}$. So we introduce the following helpful notation for all $m \in \mathbb{N}_{>0}$:

- $A_m = A/t^m A$.
- $M_m = M/t^m M$.
- $\iota_j : t^{j+1} M_m \to t^j M_m$ the natural inclusion $\forall\ j$.
- $\mu_j : t^j M_m \to t^{m-1} M_m$ the multiplication by t^{m-1-j} $\forall\ j$.
- $E^i_m(-)$ the contravariant functor $\operatorname{Ext}^i_{A_m}(-, A_m)$ $\forall\ i$.

Remark

A lemma of Rees implies that $E^i_m(M_k) \cong \operatorname{Ext}^{i+1}_A(M_k, A)$ whenever $k \leq m$. Hence we deduce that

$$E^i_m(M_k) \cong E^i_m(M_m) \ \ \forall \ \ k \leq m.$$

Fiber-full modules and flatness

Remark

Since t is a non-zero-divisor on M we have that:

$$M_j \cong t^{m-j} M_m \ \ \forall j.$$

Remark

The short exact sequences $0 \to t^{j+1} M_m \xrightarrow{\iota_j} t^j M_m \xrightarrow{\mu_j} t^{m-1} M_m \to 0$, if M is fiber-full, yield the following short exact sequences for all $i \in \mathbb{Z}$:

$$0 \to E^i_m(t^{m-1} M_m) \xrightarrow{E^i_m(\mu_j)} E^i_m(t^j M_m) \xrightarrow{E^i_m(\iota_j)} E^i_m(t^{j+1} M_m) \to 0.$$

Indeed, up to the above identifications, μ_j corresponds to the natural projection $M_{m-j} \to M_1$, therefore the map $E^i_m(\mu_j)$ is injective for all $i \in \mathbb{Z}$ by definition of fiber-full module.

Theorem

With the above notation, if M is a fiber-full A-module, then $\mathrm{Ext}^i_A(M, A)$ is flat over $K[t]$ for all $i \in \mathbb{Z}$.

Proof: By the previous lemma, it is enough to show that $E^i_m(M_m)$ is flat over $K[t]/(t^m)$ for all $m \in \mathbb{N}_{>0}$. This is clear for $m = 1$ (because $K[t]/(t)$) is a field), so we will proceed by induction: thus let us fix $m \geq 2$ and assume that $E^i_{m-1}(M_{m-1})$ is flat over $K[t]/(t^{m-1})$. The local flatness criterion tells us that is enough to show the following two properties:

1. $E^i_m(M_m)/t^{m-1}E^i_m(M_m)$ is flat over $K[t]/(t^{m-1})$.
2. The map $\theta : (t^{m-1})/(t^m) \otimes_{K[t]/(t^m)} E^i_m(M_m) \to t^{m-1}E^i_m(M_m)$ sending $\overline{t^{m-1}} \otimes \phi$ to $t^{m-1}\phi$, is a bijection.

Fiber-full modules and flatness

By the previous Remark $E^i_m(\iota_k)$ is surjective for all k, so $E^i_m(\iota^j)$ is surjective where $\iota^j := \iota_j \circ \ldots \iota_{m-2} : t^{m-1}M_m \to t^j M_m$. Since $\iota^j \circ \mu_j$ is the multiplication by t^{m-1-j} on $t^j M_m$, we therefore have

$$\mathrm{Im}(E^i_m(\mu_j)) = \mathrm{Im}(E^i_m(\mu_j) \circ E^i_m(\iota^j)) = t^{m-1-j} E^i_m(t^j M_m).$$

Therefore $\mathrm{Ker}(E^i_m(\iota_j)) = t^{m-1-j} E^i_m(t^j M_m)$. Hence

$$E^i_m(t^{j+1} M_m) \cong \frac{E^i_m(t^j M_m)}{t^{m-1-j} E^i_m(t^j M_m)}.$$

Plugging in $j = 0$, we get that

$$\frac{E^i_m(M_m)}{t^{m-1} E^i_m(M_m)} \cong E^i_m(tM_m) \cong E^i_m(M_{m-1}) \cong E^i_{m-1}(M_{m-1})$$

is flat over $K[t]/(t^{m-1})$ by induction, and this shows (1).

Fiber-full modules and flatness

Concerning (2), from what said above it is not difficult to infer that the kernel of the surjective map $E^i_m(\iota^0) : E^i_m(M_m) \to E^i_m(t^{m-1}M_m)$ is equal to $tE^i_m(M_m)$. Since $E^i_m(\mu_0) \circ E^i_m(\iota^0)$ is the multiplication by t^{m-1} on $E^i_m(M_m)$ and $E^i_m(\mu_0)$ is injective, we get

$$0 :_{E^i_m(M_m)} t^{m-1} = \mathrm{Ker}(E^i_m(\iota^0)) = tE^i_m(M_m).$$

Since $\mathrm{Ker}(\theta) = \{\overline{t^{m-1}} \otimes \phi : \phi \in 0 :_{E^i_m(M_m)} t^{m-1}\}$, then θ is injective, and so bijective (it is always surjective). □

Corollary

Let $I \subset R = K[X_1, \ldots, X_n]$ be an ideal such that $S = P/\operatorname{hom}_w(I)$ is a fiber-full P-module ($P = R[t]$). Then $\mathrm{Ext}^i_P(S, P)$ is a flat $K[t]$-module. So, if furthermore I is homogeneous:

$$\dim_K(H^i_{\mathfrak{m}}(R/I)_j) = \dim_K(H^i_{\mathfrak{m}}(R/\operatorname{in}_w(I))_j) \ \ \forall\ i, j \in \mathbb{Z}.$$

Squarefree monomial ideals and fiber-full modules

Our next goal is to show that, for an ideal $I \subset R$ such that $\mathrm{in}_w(I)$ is a squarefree monomial ideal, then $S = P/\hom_w(I)$ is a fiber-full P-module. To do so, we need to recall some notion:

Let $J \subset R$ a monomial ideal minimally generated by monomials $\mu_1, \ldots, \mu_r$. For all subset $\sigma \subset \{1, \ldots, r\}$ we define the monomial $\mu(J,\sigma) := \mathrm{lcm}(\mu_i | i \in \sigma) \in R$. If v is the qth element of σ we set $\mathrm{sign}(v,\sigma) := (-1)^{q-1} \in K$. Let us consider the graded complex of free R-modules $F_\bullet(J) = (F_i, \partial_i)_{i=0,\ldots,r}$ with

$$F_i := \bigoplus_{\substack{\sigma \subset \{1,\ldots,r\} \\ |\sigma| = i}} R(-\deg \mu(J,\sigma)),$$

and differentials defined by $1_\sigma \mapsto \sum_{v\in\sigma} \mathrm{sign}(v,\sigma) \frac{\mu(J,\sigma)}{\mu(J,\sigma\setminus\{v\})} \cdot 1_{\sigma\setminus\{v\}}$. It is well known and not difficult to see that $F_\bullet(J)$ is a graded free R-resolution of R/J, called the *Taylor resolution*.

Squarefree monomial ideals and fiber-full modules

For any positive integer k we introduce the monomial ideal

$$J^{[k]} = (\mu_1^k, \ldots, \mu_r^k).$$

Notice that $\mu_1^k, \ldots, \mu_r^k$ are the minimal system of monomial generators of $J^{[k]}$, so $\mu(J^{[k]}, \sigma) = \mu(J, \sigma)^k$ for any $\sigma \subset \{1, \ldots, r\}$.

Theorem

If $J \subset R$ is a squarefree monomial ideal, for all $i \in \mathbb{Z}$ and $k \in \mathbb{N}_{>0}$ the map $\mathrm{Ext}^i_R(R/J^{[k]}, R) \to \mathrm{Ext}^i_R(R/J^{[k+1]}, R)$, induced by the projection $R/J^{[k+1]} \to R/J^{[k]}$, is injective.

Corollary

Let $I \subset R$ be an ideal such that $\text{in}_w(I)$ is a squarefree monomial ideal. Then $S = P/\hom_w(I)$ is a fiber-full P-module.

Proof of the corollary: Notice that $\hom_w(I) + tP = \text{in}_w(I) + tP$ is a squarefree monomial ideal of P. So, by the previous theorem, the maps $\text{Ext}^i_P(S/tS, P) \to \text{Ext}^i_P(P/(\hom_w(I) + tP)^{[m]}, P)$ are injective for all $m \in \mathbb{N}_{>0}$. Since $(\hom_w(I) + tP)^{[m]} \subset \hom_w(I) + t^m P$, these maps factor through $\text{Ext}^i_P(S/tS, P) \to \text{Ext}^i_P(S/t^m S, P)$, hence the latter are injective as well. $\square$

Proof of the theorem: Let $u_1, \dots, u_r$ be the minimal monomial generators of J. For all $k \in \mathbb{N}_{>0}, \sigma \subset \{1, \dots, r\}$ set $\mu_\sigma[k] := \mu(J^{[k]}, \sigma)$ and $\mu_\sigma := \mu_\sigma[1]$. Of course μ_σ is a squarefree monomial and, for what we said above, $\mu_\sigma[k] = \mu_\sigma^k$.

The module $\mathrm{Ext}^i_R(R/J^{[k]}, R)$ is the ith cohomology of the complex $G^\bullet[k] = \mathrm{Hom}_R(F_\bullet[k], R)$ where $F_\bullet[k] = F_\bullet(J^{[k]}) = (F_i, \partial_i[k])_{i=0,\dots,r}$ is the Taylor resolution of $R/J^{[k]}$. Let $F_i \xrightarrow{f_i} F_i$ be the map sending 1_σ to $\mu_\sigma \cdot 1_\sigma$. The collection $F_\bullet[k+1] \xrightarrow{f_\bullet=(f_i)_i} F_\bullet[k]$ is a morphism of complexes lifting $R/J^{[k+1]} \to R/J^{[k]}$ (since $\mu_\sigma[k] = \mu_\sigma^k$).

So the maps $\mathrm{Ext}^i_R(R/J^{[k]}, R) \to \mathrm{Ext}^i_R(R/J^{[k+1]}, R)$ we are interested in are the homomorphisms $H^i(G^\bullet[k]) \xrightarrow{\overline{g^i}} H^i(G^\bullet[k+1])$ induced by $g^\bullet = \mathrm{Hom}(f_\bullet, R) : G^\bullet[k] \to G^\bullet[k+1]$. Let us see how $\overline{g^i}$ acts: if $G^\bullet[k] = (G^i, \partial^i[k])$, then $G^i = \mathrm{Hom}_R(F_i, R)$ can be identified with F_i (ignoring the grading) and $\partial^i[k] : G^i \longrightarrow G^{i+1}$ sends 1_σ to $\sum_{v\in\{1,\dots,r\}\setminus\sigma} \mathrm{sign}(v, \sigma\cup\{v\}) \left(\frac{\mu_{\sigma\cup\{v\}}}{\mu_\sigma}\right)^k \cdot 1_{\sigma\cup\{v\}}$ for all $\sigma \subset \{1,\dots,r\}$ and $|\sigma| = i$. The map $g^i : G^i \to G^i$, up to the identification $F_i \cong G_i$, is then the map sending 1_σ to $\mu_\sigma \cdot 1_\sigma$.

Want: $\overline{g^i}$ **injective**. Let $x \in \mathrm{Ker}(\partial^i[k])$ with $g^i(x) \in \mathrm{Im}(\partial^{i-1}[k+1])$. We need to show that $x \in \mathrm{Im}(\partial^{i-1}[k])$. Let $y = \sum_\sigma y_\sigma \cdot 1_\sigma \in G^{i-1}$ such that $\partial^{i-1}[k+1](y) = g^i(x)$. We can write y_σ uniquely as $y'_\sigma + \mu_\sigma y''_\sigma$ where no monomial in $\mathrm{supp}(y'_\sigma)$ is divided by μ_σ. If

$$y' = \sum_\sigma y'_\sigma \cdot 1_\sigma, \ y'' = \sum_\sigma y''_\sigma \cdot 1_\sigma,$$

$g^i(x) = \partial^{i-1}[k+1](y) = \partial^{i-1}[k+1](y') + \partial^{i-1}[k+1](g^{i-1}(y'')) = \partial^{i-1}[k+1](y') + g^i(\partial^{i-1}[k](y''))$. Writing $z = \sum_\sigma z_\sigma \cdot 1_\sigma$ for $\partial^{i-1}[k+1](y') \in G^i$, we have

$$z_\sigma = \sum_{v \in \sigma} \mathrm{sign}(v,\sigma) \left(\frac{\mu_\sigma}{\mu_{\sigma\setminus\{v\}}}\right)^{k+1} y'_{\sigma\setminus\{v\}}.$$

Since J is squarefree and $\mu_{\sigma\setminus\{v\}}$ does not divide $y'_{\sigma\setminus\{v\}}$ for any $v \in \sigma$, μ_σ cannot divide z_σ unless it is zero. On the other hand, μ_σ must divide z_σ by the green equality. Therefore $z_\sigma = 0$, and since σ was arbitrary $z = 0$, that is: $g^i(x) = g^i(\partial^{i-1}[k](y''))$. Being $g^i : G^i \to G^i$ obviously injective, we have found $x = \partial^{i-1}[k](y'')$. $\square$

Squarefree monomial ideals and fiber-full modules

Corollary

Let $I \subset R$ be an ideal such that $\text{in}_w(I) \subset R$ is a squarefree monomial ideal. Then $\text{Ext}^i_P(S, P)$ is a flat $K[t]$-module. So, if I is homogeneous and $\text{in}(I)$ is a squarefree monomial ideal, then

$$\dim_K(H^i_{\mathfrak{m}}(R/I)_j) = \dim_K(H^i_{\mathfrak{m}}(R/\,\text{in}(I))_j) \;\; \forall\; i, j \in \mathbb{Z}.$$

This is the arrival point for these lectures, but it suggests also some open questions...

Open questions

During the lectures we proved the following:

Theorem

Let $I \subset R$ be an ideal such that $S = P/\hom_w(I)$ is a fiber-full P-module. Then $\mathrm{Ext}^i_P(S, P)$ is a flat $K[t]$-module. So, if furthermore I is homogeneous:

$$\dim_K(H^i_{\mathfrak{m}}(R/I)_j) = \dim_K(H^i_{\mathfrak{m}}(R/\operatorname{in}_w(I))_j) \ \ \forall\ i, j \in \mathbb{Z}.$$

After this we proved that, if $\operatorname{in}_w(I) \subset R$ is a squarefree monomial ideal, then $S = P/\hom_w(I)$ is a fiber-full P-module. However, this is not the only instance: e.g., if $R/\operatorname{in}_w(I)$ is Cohen-Macaulay (equivalently if S is CM), then it is not difficult to see that S is fiber-full.

Open questions

More interestingly, we have:

- If K has positive characteristic, then S is fiber-full whenever $R/\operatorname{in}_w(I)$ is F-pure (Ma).
- If K has characteristic 0, then S is fiber-full whenever $R/\operatorname{in}_w(I)$ is Du Bois (Ma-Schwede-Shimomoto).
- S is fiber-full whenever $R/\operatorname{in}_w(I)$ is cohomologically full (a notion recently introduced by Dao-De Stefani-Ma).

Let us recall that, for a homogeneous ideal $J \subset R$, R/J is *cohomologically full* if, whenever $H \subset I$ such that $\sqrt{H} = \sqrt{J}$, the natural map $H^i_{\mathfrak{m}}(R/H) \to H^i_{\mathfrak{m}}(R/J)$ is surjective for all i.

Open questions

Often $\mathrm{in}_w(I)$ is not a monomial ideal, rather a binomial ideal s.t.

$$R/\,\mathrm{in}_w(I) \cong K[\mathcal{M}] := K[Y^u : u \in \mathcal{M}] \subset K[Y_1, \ldots, Y_m].$$

for some monoid $\mathcal{M} \subset \mathbb{N}^m$. This is the case when dealing with SAGBI (or Khovanskii) bases.

Problem

Find a big class of monoids $\mathcal{M} \subset \mathbb{N}^m$ such that $K[\mathcal{M}]$ is cohomologically full.

For example, if K has characteristic 0 and $\mathcal{M}$ is seminormal, then $K[\mathcal{M}]$ is Du Bois combining results of Bruns-Li-Römer and Schwede. So $K[\mathcal{M}]$ is cohomologically full for a seminormal monoid $\mathcal{M}$.

Open questions

We proved that, if A is a Noetherian graded flat $K[t]$-algebra, M is a f.g. A-module which is graded and flat over $K[t]$, then $\operatorname{Ext}^i_A(M,A)$ is flat over $K[t]$ for all $i \in \mathbb{Z}$ whenever M is fiber-full.

Problem

With the above notation, when is it true that $\operatorname{Ext}^i_A(M,A)$ is fiber-full whenever M is fiber-full?

For example, together with D'Alì we proved that, if M/tM is a squarefree R-module then M is fiber-full, and this implies a positive answer to the above problem when M/tM is a squarefree R-module. A consequence of this, is that the homological degrees (a notion introduced by Vasconcelos) of R/I and $R\,\mathrm{in}(I)$ are the same provided that $\mathrm{in}(I)$ is a squarefree monomial ideal.