LOCAL COHOMOLOGY, CONNECTEDNESS AND INITIAL IDEALS

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$$I \text{ prime} \Rightarrow S/LT_\prec(I) \text{ connected in codimension } 1$$

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- Connectedness of an ideal versus connectedness of its initial ideals

PART 1

LOCAL COHOMOLOGY AND CONNECTEDNESS

Notation, definitions and "basic" results

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$$\mathrm{ht}(\mathfrak{a}) \leq \mathrm{cd}(R;\mathfrak{a}) \leq \dim R.$$

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If T is irreducible, then $\mathrm{c}(T) = \dim T$.

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EXAMPLES

$$\mathfrak{a} = (xz, xw, yz, yw) \subseteq \mathbb{C}[x, y, z, w],\ T = \mathcal{Z}(\mathfrak{a}) \subseteq \mathbb{A}^4.$$

$$T = \mathcal{Z}(x, y) \cup \mathcal{Z}(z, w), \text{ and } \mathcal{Z}(x, y) \cap \mathcal{Z}(z, w) = \{(0, 0, 0, 0)\},$$

$$\text{so } \dim T = 2 \text{ and } \mathrm{c}(T) = 0.$$

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If $T = \operatorname{Spec} R$, then $\mathrm{c}(R) := \mathrm{c}(T)$ is the infimum of $\dim R/\mathfrak{a}$ with $\mathfrak{a}$ ideal such that $\operatorname{Spec} R \setminus \mathcal{V}(\mathfrak{a})$ is disconnected.

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PART 2

APPLICATIONS TO INITIAL IDEALS

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- $\omega = (\omega_1, \dots, \omega_n) \in (\mathbb{Z}_+)^n$ weight vector;

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- $f \in S$, $\mathrm{in}_\omega(f) \in S$ is the leading coefficient of

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For every ω and I, if I is prime, then

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Choose ω such that $\operatorname{in}_\omega(I) = LT_\prec(I)$.

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Follows from the fact that $\mathrm{c}(S/I) \geq \mathrm{depth}(S/I) - 1$.

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Graded version of the main result of the first part let us conclude!

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So S/J is not connected in codimension 1, therefore

cannot exist $I \subseteq S$ Cohen-Macaulay and $\prec$ s. t. $\sqrt{LT_{\prec}(I)} = J$!

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In this case we have also that $\mathrm{cd}(S, I) < \mathrm{cd}(S, LT_{\prec}(I))$!

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Notice that there are squarefree monomial ideals connected in codimension one with h-vector with negative coefficients, so they cannot be the initial ideal of any Cohen-Macaulay ideal.

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Then, for every term order respecting the total degree,

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