LYUBEZNIK NUMBERS AND DEPTH

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Bass numbers

Let R be a noetherian ring and M an R-module. Consider a minimal injective resolution:

$$0 \to M \to E^0 \to E^1 \to E^2 \to \dots$$

The indecomposable injective R-modules are $E_R(R/\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec} R$. The Bass numbers of M are defined as the number $\mu_i(\mathfrak{p}, M)$ of copies of $E_R(R/\mathfrak{p})$ occurring in E^i. In other words:

$$E^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec}(R)} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)}.$$

Theorem: $\mu_i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$, where $\kappa(\mathfrak{p}) = \dfrac{R_\mathfrak{p}}{\mathfrak{p} R_\mathfrak{p}}$.

In particular, if M is finitely generated, each Bass number is finite.

Local cohomology modules

Throughout the talk all the rings we consider are noetherian.

Given an ideal I of an n-dimensional ring S and an S-module M, we will freely use the following facts on the local cohomology modules $H_I^i(M)$ (which are not finitely generated in general):

(i) (Grothendieck): $H_I^i(M) = 0$ if $i > \mathrm{Supp}(M)$. If M is finitely generated, then $H_I^i(M) = 0$ whenever $i < \mathrm{grade}(I, M)$.

(ii) (Hartshorne-Lichtenbaum) $\Rightarrow$ If $(S, \mathfrak{m})$ is regular local, then

$$H_I^n(S) = 0 \Leftrightarrow \dim S/I > 0$$

(iii) (Peskine-Szpiro, Ogus, Huneke-Lyubeznik) $\Rightarrow$ If $(S, \mathfrak{m})$ is regular, contains a field and $\mathrm{depth}(S/I) \geq 2$, then

$$H_I^{n-1}(S) = H_I^n(S) = 0$$

Lyubeznik numbers

Theorem (Huneke-Sharp, Lyubeznik): If $(S, \mathfrak{m})$ is a regular local ring containing a field, $I \subseteq S$ is any ideal and j is any natural number, then each Bass number of $H_I^j(S)$ is finite.

Definition-Theorem (Lyubeznik): Let $(R, \mathfrak{n})$ be local containing a field. The completion $\widehat{R}$ is isomorphic to S/I, where $I \subseteq S = k[[x_1, \ldots, x_n]]$. The Bass numbers $\mu_i(\mathfrak{m}, H_I^{n-j}(S))$ depend only on R, i and j. The Lyubeznik numbers of R are therefore defined as:

$$\lambda_{i,j}(R) = \mu_i(\mathfrak{m}, H_I^{n-j}(S)).$$

He also showed that $H_{\mathfrak{m}}^i(H_I^{n-j}(S)) \cong E_S(k)^{\lambda_{i,j}(R)}$.

Basic properties

For a while, $(R, \mathfrak{n})$ will be a local ring containing a field k, $S = k[[x_1, \ldots, x_n]]$, $\mathfrak{m} = (x_1, \ldots, x_n)$ and $I \subseteq S$ s. t. $\widehat{R} \cong S/I$.

If $\dim R = d$, then $I \subseteq S$ has height $n - d$. In particular $H_I^{n-j}(S)$ vanishes whenever $j > d$, therefore:

$$\lambda_{i,j}(R) = 0 \qquad \forall\ j > d.$$

If $\mathfrak{p}$ is a height $c < n - j$ prime of S, then

$$H_I^{n-j}(S)_{\mathfrak{p}} = H_{IS_{\mathfrak{p}}}^{n-j}(S_{\mathfrak{p}}) = 0.$$

So $\dim \mathrm{Supp}(H_I^{n-j}(S)) \leq j$. In particular, $H_{\mathfrak{m}}^i(H_I^{n-j}(S)) = 0$ whenever $i > j$, so that:

$$\lambda_{i,j}(R) = 0 \qquad \forall\ i > j.$$

The Lyubeznik table

Thus the following $(d+1) \times (d+1)$ upper triangular matrix is an invariant of a d-dimensional local ring R containing a field:

$$\Lambda(R) = \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} & \lambda_{0,2} & \cdots & \lambda_{0,d} \\ 0 & \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,d} \\ 0 & 0 & \lambda_{2,2} & \cdots & \lambda_{2,d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_{d,d} \end{pmatrix}$$

.

(here $\lambda_{i,j} = \lambda_{i,j}(R)$). The above matrix is pretty mysterious, however there are various results describing some of the entries...

Easy statements

(i) Suppose that the ideal I for which $\widehat{R} \cong S/I$ has minimal primes of height at most $n-b$ (therefore $b \leq d$). Then:

$$\lambda_{i,i}(R) = 0 \quad \forall\ i < b.$$

Since $\sqrt{IS_{\mathfrak{p}}} \neq \mathfrak{p}S_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Spec}(S)$ of height $n-i$, we have $\dim \mathrm{Supp}(H_I^{n-i}(S)) < i$ for all $i < b$. So $H_{\mathfrak{m}}^i(H_I^{n-i}(S)) = 0$.

(ii) If R is a complete intersection, then $H_I^{n-j}(S) = 0$ for all $j < d$ because I is generated by $n-d$ elements. So $\lambda_{i,j}(R) = 0$ if $j < d$. Furthermore, because the second page of the spectral sequence

$$E_2^{i,j} = H_{\mathfrak{m}}^i(H_I^{n-j}(S)) \Rightarrow H_{\mathfrak{m}}^{n+i-j}(S)$$

is full of zeroes, it is easy to infer that $\lambda_{i,d}(R) = \delta_{i,d}$.

More serious results

Theorem (Zhang): $\lambda_{d,d}(R)$ is the number of connected components of the codimension 1 graph of $(S/I) \otimes_k \overline{k}$.

Theorem (Blickle-Bondu): If $R_{\mathfrak{p}}$ is a complete intersection for any prime ideal $\mathfrak{p} \neq \mathfrak{n}$, then $\lambda_{i,d}(R) - \delta_{i,d} = \lambda_{0,d-i+1}(R)$ and $\lambda_{i,j}$ vanishes whenever $0 < i$ and $j < d$.

Theorem (Garcia Lopez-Sabbah, Blickle-Bondu, Blickle): If, besides satisfying the condition above, $R = \mathcal{O}_{Y,y}$ for a closed k-subvariety of a smooth variety X, then

$$\lambda_{0,j}(R) = \begin{cases} \dim_{\mathbb{C}} H^j_{\{y\}}(Y_{\text{an}}, \mathbb{C}) & \text{if } k = \mathbb{C} \\ \dim_{\mathbb{Z}/p\mathbb{Z}} H^j_{\{y\}}(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) & \text{if } k = \mathbb{Z}/p\mathbb{Z} \end{cases}$$

Geometric invariant?

Conjecture (Lyubeznik): Let X be a projective scheme over k. The Lyubeznik table of the coordinate ring of X (localized at the maximal irrelevant) is actually an invariant of X.

All the previous results provide evidence for the above conjecture.

(Zhang): True in positive characteristic!

Vanishing of $\lambda_{i,j}$ from the depth

Proposition: $\lambda_{i,j}(R) = 0$ for all $j < \mathrm{depth}(R)$ and $i \geq j - 1$.

Proof: If we pick a prime $\mathfrak{p} \in \mathrm{Spec}(S)$ of height $n - j + 1$, then:

$$\mathrm{Ext}^r_S(S/\mathfrak{p}, R) = 0 \quad \forall\ r < \mathrm{depth}(R) - j + 1 \leq 2.$$

In particular $\mathrm{depth}(S_\mathfrak{p}/IS_\mathfrak{p}) \geq 2$, which thereby implies

$$H^{n-j}_I(S)_\mathfrak{p} \cong H^{n-j}_{IS_\mathfrak{p}}(S_\mathfrak{p}) = 0$$

so that $\dim \mathrm{Supp}(H^{n-j}_I(S)) < j - 1$. $\square$

Vanishing of $\lambda_{i,j}$ from the depth

Notice that, if $\mathrm{char}(k) > 0$, then $\lambda_{i,j}(R) = 0$ for all $j < \mathrm{depth}(R)$.

Proof: Peskine-Szpiro $\Rightarrow H_I^{n-j}(S) = 0 \;\; \forall \, j < \mathrm{depth}(S/I)$. $\square$

That is false in characteristic 0: consider $R = (k[X]/I_t(X))_{(X)}$ where X is an $m \times n$-matrix. By Bruns-Schwänzl:

$$\lambda_{i,j}(R) = \begin{cases} 0 & \text{if } j < t^2 - 1 \\ 0 & \text{if } j = t^2 - 1 \text{ and } i > 0 \\ 1 & \text{if } j = t^2 - 1 \text{ and } i = 0 \\ ??? & \text{otherwise} \end{cases}$$

But R is Cohen-Macaulay of dimension $(t-1)(m+n-t+1)$.

Vanishing of $\lambda_{i,j}$ from the depth

Conjecture: $\lambda_{i,j}(R) = 0 \quad \forall\, j < \mathrm{depth}(R)$ and $i \geq j - 2$.

For example, according to this conjecture, the Lyubeznik table of a 7-dimensional local ring of depth 6 should look like:

$$\Lambda(R) = \begin{pmatrix} 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

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Vanishing from the depth

Proposition: The above conjecture is equivalent to show the following: If I is an ideal in an n-dimensional regular local ring S containing a field such that $\mathrm{depth}(S/I) \geq 3$, then:

$$H_I^{n-2}(S) = H_I^{n-1}(S) = H_I^{n}(S) = 0.$$

Proof: $\Rightarrow$: In any case $H_I^{n-2}(S)$ is supported only at the maximal ideal, so $H_I^{n-2}(S) \cong E(k)^s$ (Lyubeznik), so $\lambda_{0,2}(S/I) = s$. For the converse implication argue like in the proof of few slides above. $\square$

Vanishing from the depth

Theorem (-): If $S = k[x_1, \dots, x_n]$ and $I \subseteq S$ is homogeneous such that $\mathrm{depth}(S/I) \geq 3$, then

$$H_I^{n-2}(S) = H_I^{n-1}(S) = H_I^n(S) = 0$$

Sketch of the proof: After some manipulations we see, thanks to a result of Ogus, that we need to show that $H^2_{\mathrm{DR},\mathfrak{m}}(\mathrm{Spec}(S/I)) = 0$. In our setting, from a work of Harthorne this is equivalent to prove:

$$H^1(X_{\mathrm{an}}, \mathbb{C}) = 0,$$

where $X = \mathrm{Proj}(S/I)$ and we are assuming $k = \mathbb{C}$. The universal coefficient theorem and the exponential sequence make the job:

$$H^1(X_{\mathrm{an}}, \mathbb{Z}) \hookrightarrow H^1(X_{\mathrm{an}}, \mathcal{O}_{X_{\mathrm{an}}}) \cong H^2_{\mathfrak{m}}(S/I)_0 = 0.$$

$\square$

Set-theoretically Cohen-Macaulayness

Corollary: Let $X \subseteq \mathbb{P}^{n-1}$ be a smooth surface with irregularity $q(X) > 0$ over a field of characteristic 0. Then for any graded ideal $I \subseteq S$ defining X set-theoretically S/I is not Cohen-Macaulay.

Proof: One can show that $H_I^{n-2}(S)$ (which does not depend on the radical of I!) is not zero. $\square$

Question: Are there analog examples for connected curves? Can you find a graded ideal $I \subseteq \mathbb{C}[a, b, c, d]$ defining set-theoretically $X = \{[s^4, s^3t, st^3, t^4] : [s, t] \in \mathbb{P}^1\} \subseteq \mathbb{P}^3$ such that $\mathbb{C}[a, b, c, d]/I$ is Cohen-Macaulay?

THANK YOU !!!