SYMBOLIC POWERS AND MATROIDS

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SKETCH OF THE PROOF

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Using a theorem of *Eisenbud* and *Huneke*, the above result yields

$$\dim(\Delta) + 1 = \mathrm{ht}(J(\Delta)) \leq \dim(\bar{A}(\Delta)) = n - \min_{k \in \mathbb{N}_{>0}} \{\mathrm{depth}(S/J(\Delta)^{(k)})\}.$$

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Our aim, thus, is to show that $\dim(\bar{A}(\Delta)) = \dim(\Delta) + 1$ whenever Δ is a matroid.

IF Δ IS A MATROID...

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which, from the Hilbert polynomial, gets $\dim(\bar{A}(\Delta)) = d$.

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Therefore (I) and (II) together yield $\alpha(j_0) = \alpha(i_0)$.

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Hence $S/J(\Delta)^{(k)}$ and $S/I_\Delta^{(k)}$ are Cohen-Macaulay for any $k \in \mathbb{N}_{>0}$!

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Eventually, consider $\mathcal{H} := \widetilde{J(\Delta)^{(d+1)}}$. $(d + 1 + d - 1 = 2d)$.

By the previous slide, $\wp_{F,\mathbf{a}}$ and $\wp_{G,\mathbf{b}}$ belong to $\mathrm{Ass}(\mathcal{H})$, where

$\mathbf{a} := (d+1, 1, \ldots, 1) \in \mathbb{N}^d$ and $\mathbf{b} := (2, 2, \ldots, 2) \in \mathbb{N}^d$ $(|\mathbf{a}| = |\mathbf{b}| = 2d)$.

We will show that $R := \widetilde{S}/\mathcal{H}$ is not Cohen-Macaulay:

Were it, $R_{\wp_{F,\mathbf{a}}+\wp_{G,\mathbf{b}}}$ would be Cohen-Macaulay too.

Particularly, $R_{\wp_{F,\mathbf{a}}+\wp_{G,\mathbf{b}}}$ would be connected in codimension 1.

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$$\wp = (x_{i_s,a_s}, \quad x_{j_q,b_q} \ : \ s \in [d] \setminus \{p\}).$$

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So, there should be a prime $\wp \in \mathrm{Ass}(\mathcal{H})$ such that:

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So, Δ has to be a matroid!

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