Determinantal facet ideals

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Notation

Let $X = (X_{ij})$ be a $r \times n$ generic matrix with $r \leq n$, and set $S := K[X]$ over a field K. Given $1 \leq i_1 < i_2 < \ldots < i_r \leq n$ we set

$$[i_1 i_2 \ldots i_r] := \det \begin{pmatrix} X_{1i_1} & \ldots & X_{1i_r} \\ \vdots & & \vdots \\ X_{ri_1} & \ldots & X_{ri_r} \end{pmatrix} \in S$$

Definition

Let $\mathcal{M}_r(X) := \{[i_1 i_2 \ldots i_r] : 1 \leq i_1 < i_2 < \ldots < i_r \leq n\}$ be the set of r-**minors** of X.

Determinantal facet ideals

Definition

The **determinantal facet ideal** of a subset $C \subseteq \mathcal{M}_r(X)$ is

$$J_C := (C) \subseteq S$$

Remark

The name comes from the fact that the elements of C are (identifying $[i_1 \dots i_r]$ with $\{i_1, \dots, i_r\}$) the facets of the simplicial complex $\langle C \rangle = \{\sigma \subseteq \tau : \tau \in C\}$.

Notice that there is the following 1-1 correspondence:

$$\mathcal{Z}(J_C) \subseteq \mathbb{A}_K^{r \times n} \quad \leftrightarrow \quad \left\{ \begin{array}{c} \text{realization spaces over } K \text{ of matroids on } [n] \\ \text{of rank} \leq r \text{ with } C \subseteq \{\text{dependent sets}\} \end{array} \right\}$$

In general, J_C is not radical:

Example

If $r = 3$, $n = 5$ and $C = \{[124], [145], [234], [345]\}$, one can check that J_C is not radical.

In the case $r = 2$, determinantal facet ideals are known as **binomial edge ideals**. They were independently introduced by *Ohtani* and by *Herzog, Hibi, Hreinsdóttir, Kahle, Rauh*. Among other things they proved that they are radical. More precisely, they admit a squarefree initial ideal. They also described combinatorially the set of minimal primes for any binomial edge ideal.

F-singularities of binomial edge ideals

Since binomial edge ideals admit a squarefree initial ideal:

Fact

If $\mathrm{char}(K) > 0$, then S/J_C is F-injective[a] whenever $r = 2$.

[a] I.e. the Frobenius acts injectively on $H^i_{(X)}(S/J_C)\ \forall\ i \in \mathbb{N}$

Theorem (Matsuda, 2018)

If $\mathrm{char}(K) > 0$ and $r = 2$, then S/J_C is F-pure[a] whenever C is weakly closed (aka co-comparable)[b].

[a] I.e. the Frobenius map on S/J_C is pure

[b] $[ik] \in C \implies [ij] \in C$ or $[jk] \in C$ whenever $i < j < k$.

F-singularities of binomial edge ideals

When $r = 2$:

Conjecture (Matsuda, 2018)

S/J_C is F-pure whenever $\mathrm{char}(K) \gg 0$.

Conjecture (Matsuda, 2018)

If $\mathrm{char}(K) = 2$, then S/J_C is F-pure $\iff$ C is co-comparable (up to shuffling the columns of X).

F-singularities of binomial edge ideals - Example

Let $r = 2, n = 5$, $C = \{[12], [23], [34], [45], [15]\}$.

Remark

Notice that C is not co-comparable: $[15] \in C$, $[13], [35] \notin C$. No shuffling of the columns will give co-comparability for C.

In this case:

- If $\mathrm{char}(K) = 2$, then S/J_C is *not* F-pure.
- If $\mathrm{char}(K) = 3, 5, 7$, then S/J_C is F-pure.
- Is S/J_C F-pure whenever $\mathrm{char}(K) > 2$?

Knutson ideals

Fix a monomial order $<$ on S. Let $f \in S$ be such that $\text{in}(f)$ is a squarefree monomial.

Definition

We denote by $\mathcal{C}_f$ the set of ideals of S built following the rules:

- $(f) \in \mathcal{C}_f$.
- $I \in \mathcal{C}_f \implies I : J \in \mathcal{C}_f \quad \forall\ J \subseteq S$.
- $I, J \in \mathcal{C}_f \implies I + J,\ I \cap J \in \mathcal{C}_f$.

The ideals in $\mathcal{C}_f$ are called **Knutson ideals**. In 2009 *Allen Knutson* proved remarkable facts about ideals in $\mathcal{C}_f$ in the case $K = \mathbb{Z}/p\mathbb{Z}$. The following has been generalized to any field K by *Lisa Seccia*:

Theorem (Knutson 2009, Seccia, 2021)

If $I \in \mathcal{C}_f$, then:

- $\text{in}(I)$ (and so I) is radical.
- If $\text{char}(K) > 0$, then S/I is F-pure.

Knutson ideals

Lemma

If I and J are ideals of S, TFAE:

- $\text{in}(I + J) = \text{in}(I) + \text{in}(J)$.
- $\text{in}(I \cap J) = \text{in}(I) \cap \text{in}(J)$.

If $\text{in}(I \cap J)$ is radical, then $\text{in}(I \cap J) = \text{in}(I) \cap \text{in}(J)$.

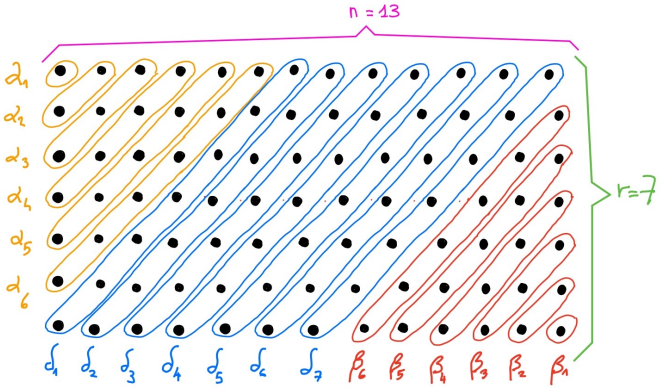
This simple lemma implies a surprising fact:

Corollary

If I and J are ideals in C_f, then $\text{in}(I + J) = \text{in}(I) + \text{in}(J)$. In other words, the union of the Gröbner bases of I and J is a Gröbner basis of the sum $I + J$.

Ex: the ideal of maximal minors

Let $I_r = I_r(X) \subseteq S$ the ideal of maximal minors of X. Then I_r is a prime ideal of height $n - r + 1$, and contains the complete intersection $C = (\delta_1, \ldots, \delta_{n-r+1}) \subseteq S$, where the δ_i's are the maximal minors insisting on consecutive columns of X:



Ex: the ideal of maximal minors of a generic matrix

Put $f = \prod_{i=1}^{n-r+1} \delta_i \in S$. If $<$ is the degrevlex term order with

$$X_{11} > X_{12} > \ldots > X_{1n} > X_{21} > \ldots > X_{2n} > \ldots > X_{rn},$$ [1]

then $\text{in}(f) = \prod_{i=1}^{n-r+1} \text{in}(\delta_i)$, and $\text{in}(\delta_i) = \pm \prod_{a+b=r+i} X_{ab}$. In particular, $\text{in}(f)$ is squarefree.

As factors of f, the principal ideals generated by the δ_i's are in $\mathcal{C}_f$. So $C = (\delta_1, \ldots, \delta_{s-r+1})$, being the sum of ideals in $\mathcal{C}_f$, is in $\mathcal{C}_f$. Since I_r is a prime ideal containing C and height $I_r =$ height C, then $I_r = C : g$ for some $g \in S$. So $I_r \in \mathcal{C}_f$ as well.

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[1] From now on we fix such a monomial order

Knutson ideals

From now on:

$$\Delta := \prod_{i=1}^{r-1} \alpha_i \prod_{i=1}^{n-r+1} \delta_i \prod_{i=1}^{r-1} \beta_i \in S.$$

Since $\mathrm{in}(\Delta) = \pm \prod_{i,j} X_{ij}$ is squarefree, it makes sense to study $\mathcal{C}_\Delta$:

Theorem (Seccia, 2021)

For any $1 \leq t \leq r$, the ideals of t-minors insisting on consecutive (resp. all) rows and all (resp. consecutive) columns are in $\mathcal{C}_\Delta$. In particular $I_t(X)$ belongs to $\mathcal{C}_\Delta$.

She also characterized all the ideals in C_Δ when $r = 2$. As a consequence:

Theorem (Seccia, 2022)

The binomial edge ideals in $\mathcal{C}_\Delta$ $(r = 2)$ are exactly those corresponding to the co-comparable collections of 2-minors.

Determinantal facet ideals - Reducedness

Definition (Benedetti, Seccia, V (2022))

A subset $C \subseteq \mathcal{M}_r(X)$ is **semi-closed** if whenever $[i_1 i_2 \dots i_r] \in C$, one of the following two conditions hold:

1. either $[i_1 a_2 \dots a_r] \in C$ whenever $a_j \leq i_j$ for all $j = 2, \dots, r$.
2. or $[z_1 \dots z_{r-1} i_r] \in C$ whenever $i_j \leq z_j$ for all $j = 1, \dots, r-1$.

By taking sums and intersections of ideals of minors insisting on consecutive columns we can prove:

Theorem (Benedetti, Seccia, V, 2022)

If C is semi-closed, then $J_C \in C_\Delta$. In particular $\mathrm{in}(J_C)$ (and so J_C) is radical and S/J_C is F-pure whenever $\mathrm{char}(K) > 0$.

Definition

A subset $C \subseteq \mathcal{M}_r(X)$ is **unit-interval** if whenever $[i_1 i_2 \dots i_r] \in C$, then $[a_1 a_2 \dots a_r] \in C$ for all $i_1 \leq a_1 < a_2 < \dots < a_r \leq i_r$.

Remark

Of course semi-closed implies unit-interval. Notice that C is unit-interval if and only if J_C is a sum of ideals of r-minors insisting on consecutive columns.

As a consequence of what previously said, if C is unit-interval then C is a Gröbner basis. Clearly C is a GB also when C is closed:

Definition (Ene, Herzog, Hibi, 2013)

A subset $C \subseteq \mathcal{M}_r(X)$ is **closed** if whenever $[i_1 i_2 \dots i_r] \in C$ and $[j_1 j_2 \dots j_r] \in C$ with $i_k = j_k$ for some k, then $[a_1 a_2 \dots a_r] \in C$ for any r-subset $\{a_1, \dots, a_r\} \subset \{i_1, \dots, i_r, j_1, \dots, j_r\}$.

Proposition (Benedetti, Seccia, V, 2022)

Let C be strongly connected. If C is closed, then C is unit-interval.

Remark

The converse is false: if $r = 3, n = 5$, then the subset of 3-minors $C = \mathcal{M}_3(X^{[1,4]}) \cup \mathcal{M}_3(X^{[2,5]})$ is unit-interval, strongly connetced but not closed (take $[134], [235] \in C$).

Theorem (Benedetti, Seccia, V, 2022)

Suppose that $\delta_i = [i \dots i + d - 1] \in C$ for all $i = 1, \dots, n - d + 1$. Then C is a Gröbner basis if and only if C is unit-interval.

Remark

Almousa and VandeBogert defined **lcm-closed** determinantal facet ideals, a combinatorial notion simultaneously generalizing "closed" and "unit-interval", and conjectured it is equivalent to C being a Gröbner basis. It can be seen that under the assumption that $\delta_i = [i \dots i + d - 1] \in C$ for all $i = 1, \dots, n - d + 1$, then the collection C is unit-interval if and only if it is lcm-closed.

Further developments

Theorem (De Stefani, Montaño, Núñez-Betancourt, Seccia, V)

For any $1 \leq t \leq r$, the ideals of t-minors contained in a top-left (resp. bottom-right) corner of the matrix X are in $\mathcal{C}_\Delta$.

Since any **matrix Schubert determinantal ideal** can be written as sum of ideals of minors contained in top-left corners of a generic matrix, they are in $\mathcal{C}_\Delta$ and the minors generating them are a Gröbner basis. The latter fact had been proved by *Knutson and Miller* in 2005.
Another interesting consequence is that matrix Schubert determinantal ideals define F-pure rings whenever $\mathrm{char}(K) > 0$, as proved by *Brion and Kumar* in 2005.
Indeed, as noticed by Knutson, a similar argument to the one used by Brion and Kumar shows that matrix Schubert ideals are in $\mathcal{C}_\Delta$.

Further developments

Denote with $\mathcal{M}(X) = \mathcal{M}_1(X) \cup \mathcal{M}_2(X) \cup \ldots \cup \mathcal{M}_r(X)$ be the set of minors of X endowed with the partial order making determinantal rings graded ASL, i.e.:

$$[a_1 \ldots a_s | b_1 \ldots b_s] \leq [c_1 \ldots c_t | d_1 \ldots d_t] \iff \begin{array}{l} s \geq t \text{ and } \forall\ i \in 1, \ldots, t \\ a_i \leq c_i \text{ and } b_i \leq d_i. \end{array}$$

Since ideals of $K[X]$ generated by any poset ideal of $\mathcal{M}(X)$ can be written as intersection of sums of ideals of minors contained in top-left corners of X:

Corollary (De Stefani, Montaño, Núñez-Betancourt, Seccia, V)

Ideals $I \subseteq K[X]$ generated by any poset ideal of $\mathcal{M}(X)$ belongs to $\mathcal{C}_\Delta$. In particular $K[X]/I$ is F-pure whenever $\text{char}(K) > 0$.

Remark

Any graded ASL over a field of positive characteristic is F-injective, but some are not F-pure.

THANKS FOR YOUR ATTENTION!