

LYUBEZNIK NUMBERS AND DEPTH

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Bass numbers

Let R be a noetherian ring and M an R -module. Consider a minimal injective resolution:

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots$$

The indecomposable injective R -modules are $E_R(R/\mathfrak{p})$ for some $\mathfrak{p} \in \text{Spec } R$. The **Bass numbers** of M are defined as the number $\mu_i(\mathfrak{p}, M)$ of copies of $E_R(R/\mathfrak{p})$ occurring in E^i . In other words:

$$E^i \cong \bigoplus_{\mathfrak{p} \in \text{Spec}(R)} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)}.$$

Theorem: $\mu_i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^i(\kappa(\mathfrak{p}), M_{\mathfrak{p}})$, where $\kappa(\mathfrak{p}) = \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}}$.

In particular, if M is finitely generated, each Bass number is finite.

Local cohomology modules

Throughout the talk all the rings we consider are noetherian.

Given an ideal I of an n -dimensional ring S and an S -module M , we will freely use the following facts on the local cohomology modules $H_i^j(M)$ (which are not finitely generated in general):

- (i) (Grothendieck): $H_i^j(M) = 0$ if $i > \dim \text{Supp}(M)$. If S is Cohen-Macaulay, then $H_i^j(S) = 0$ whenever $i < \text{ht } I$.
- (ii) (Hartshorne-Lichtenbaum) \Rightarrow If (S, \mathfrak{m}) is regular local, then

$$H_i^n(S) = 0 \Leftrightarrow \sqrt{I} \neq \mathfrak{m}$$

- (iii) (Peskin-Szpiro, Ogus, Huneke-Lyubeznik) \Rightarrow If (S, \mathfrak{m}) is regular, contains a field and $\text{depth}(S/I) \geq 2$, then

$$H_i^{n-1}(S) = H_i^n(S) = 0$$

Lyubeznik numbers

Theorem (Huneke-Sharp, Lyubeznik): If (S, \mathfrak{m}) is a regular local ring containing a field, $I \subseteq S$ is any ideal and j is any natural number, then each Bass number of $H_I^j(S)$ is finite.

Definition-Theorem (Lyubeznik): Let (R, \mathfrak{n}) be local containing a field. The completion \widehat{R} is isomorphic to S/I , where $I \subseteq S = k[[x_1, \dots, x_n]]$. The Bass numbers $\mu_i(\mathfrak{m}, H_I^{n-j}(S))$ depend only on R , i and j . The **Lyubeznik numbers** of R are therefore defined as:

$$\lambda_{i,j}(R) = \mu_i(\mathfrak{m}, H_I^{n-j}(S)).$$

He also showed that $H_{\mathfrak{m}}^i(H_I^{n-j}(S)) \cong E_S(k)^{\lambda_{i,j}(R)}$.

Basic properties

For a while, (R, \mathfrak{n}) will be a local ring containing a field k ,
 $S = k[[x_1, \dots, x_n]]$, $\mathfrak{m} = (x_1, \dots, x_n)$ and $I \subseteq S$ s. t. $\widehat{R} \cong S/I$.

If $\dim R = d$, then $I \subseteq S$ has height $n - d$. In particular $H_I^{n-j}(S)$ vanishes whenever $j > d$, therefore:

$$\lambda_{i,j}(R) = 0 \quad \forall j > d.$$

If \mathfrak{p} is a height $c < n - j$ prime of S , then

$$H_I^{n-j}(S)_{\mathfrak{p}} = H_{I_{S_{\mathfrak{p}}}}^{n-j}(S_{\mathfrak{p}}) = 0.$$

So $\dim \text{Supp}(H_I^{n-j}(S)) \leq j$. In particular, $H_{\mathfrak{m}}^i(H_I^{n-j}(S)) = 0$ whenever $i > j$, so that:

$$\lambda_{i,j}(R) = 0 \quad \forall i > j.$$

The Lyubeznik table

Thus the following $(d + 1) \times (d + 1)$ upper triangular matrix is an invariant of a d -dimensional local ring R containing a field:

$$\Lambda(R) = \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} & \lambda_{0,2} & \cdots & \lambda_{0,d} \\ 0 & \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,d} \\ 0 & 0 & \lambda_{2,2} & \cdots & \lambda_{2,d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_{d,d} \end{pmatrix}$$

(here $\lambda_{i,j} = \lambda_{i,j}(R)$). The above matrix is pretty mysterious, however there are various results describing some of the entries...

Easy statements

(i) Suppose that the ideal I for which $\widehat{R} \cong S/I$ has **minimal primes of height at most $n - b$** (therefore $b \leq d$). Then:

$$\lambda_{i,j}(R) = 0 \quad \forall i < b.$$

Since $\sqrt{IS_{\mathfrak{p}}} \neq \mathfrak{p}S_{\mathfrak{p}}$ for any $\mathfrak{p} \in \text{Spec}(S)$ of height $n - i$, we have $\dim \text{Supp}(H_I^{n-i}(S)) < i$ for all $i < b$. So $H_m^i(H_I^{n-i}(S)) = 0$.

(ii) If R is a **complete intersection**, then $H_I^{n-j}(S) = 0$ for all $j < d$ because I is generated by $n - d$ elements. So $\lambda_{i,j}(R) = 0$ if $j < d$. Furthermore, because the second page of the spectral sequence

$$E_2^{i,j} = H_m^i(H_I^{n-j}(S)) \Rightarrow H_m^{n+i-j}(S)$$

is full of zeroes, it is easy to infer that $\lambda_{i,d}(R) = \delta_{i,d}$.

More serious results

Theorem (Zhang): $\lambda_{d,d}(R)$ is the number of connected components of the codimension 1 graph of $(S/I) \otimes_k \bar{k}$.

Theorem (Blickle-Bondu): If $R_{\mathfrak{p}}$ is a complete intersection for any prime ideal $\mathfrak{p} \neq \mathfrak{n}$, then $\lambda_{i,d}(R) - \delta_{i,d} = \lambda_{0,d-i+1}(R)$ and $\lambda_{i,j}$ vanishes whenever $0 < i$ and $j < d$.

Theorem (Switala): If A is the coordinate ring of a smooth projective scheme X over \mathbb{C} , then

$$\lambda_{0,j}(A_{A_+}) = \begin{cases} \dim_{\mathbb{C}} \widetilde{H}^{j-1}(X, \mathbb{C}) - \dim_{\mathbb{C}} \widetilde{H}^{j-3}(X, \mathbb{C}) & \text{if } j \in \{1, \dots, \dim X\} \\ 0 & \text{if } j = \dim X + 1 \end{cases}$$

Geometric invariant?

Conjecture (Lyubeznik): Let X be a projective scheme over k . The Lyubeznik table of the coordinate ring of X (localized at the maximal irrelevant) is actually an invariant of X .

All the previous results provide evidence for the above conjecture.

(Zhang): True in positive characteristic!

Vanishing of $\lambda_{i,j}$ from the depth

Proposition: $\lambda_{i,j}(R) = 0$ for all $j < \text{depth}(R)$ and $i \geq j - 1$.

Proof. If we pick a prime $\mathfrak{p} \in \text{Spec}(S)$ of height $n - j + 1$, then:

$$\text{Ext}_S^r(S/\mathfrak{p}, R) = 0 \quad \forall r < \text{depth}(R) - j + 1 \geq 2.$$

In particular $\text{depth}(S_{\mathfrak{p}}/IS_{\mathfrak{p}}) \geq 2$, which thereby implies

$$H_I^{n-j}(S)_{\mathfrak{p}} \cong H_{IS_{\mathfrak{p}}}^{n-j}(S_{\mathfrak{p}}) = 0$$

so that $\dim \text{Supp}(H_I^{n-j}(S)) < j - 1$. \square

Vanishing of $\lambda_{i,j}$ from the depth

Notice that, if $\text{char}(k) > 0$, then $\lambda_{i,j}(R) = 0$ for all $j < \text{depth}(R)$.

Proof. Peskine-Szpiro $\Rightarrow H_i^{n-j}(S) = 0 \quad \forall j < \text{depth}(S/I)$. \square

That is false in characteristic 0: consider $R = (k[X]/I_t(X))_{(X)}$ where X is an $m \times n$ -matrix. By Bruns-Schwänzl:

$$\lambda_{i,j}(R) = \begin{cases} 0 & \text{if } j < t^2 - 1 \\ 0 & \text{if } j = t^2 - 1 \text{ and } i > 0 \\ 1 & \text{if } j = t^2 - 1 \text{ and } i = 0 \\ ??? & \text{otherwise} \end{cases}$$

But R is Cohen-Macaulay of dimension $(t-1)(m+n-t+1)$.

Vanishing of $\lambda_{i,j}$ from the depth

Conjecture: $\lambda_{i,j}(R) = 0 \quad \forall j < \text{depth}(R) \text{ and } i \geq j - 2.$

For example, according to this conjecture, the Lyubeznik table of a 7-dimensional local ring of depth 6 should look like:

$$\Lambda(R) = \begin{pmatrix} 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

Vanishing from the depth

Proposition: The above conjecture is equivalent to show the following: If I is an ideal in an n -dimensional regular local ring S containing a field such that $\text{depth}(S/I) \geq 3$, then:

$$H_i^{n-2}(S) = H_i^{n-1}(S) = H_i^n(S) = 0.$$

Proof. \Rightarrow : In any case $H_i^{n-2}(S)$ is supported only at the maximal ideal, so $H_i^{n-2}(S) \cong E(k)^s$ (Lyubeznik), so $\lambda_{0,2}(S/I) = s$. For the converse implication argue like in the proof of few slides above. \square

Vanishing from the depth

Theorem (-): If $S = k[x_1, \dots, x_n]$ and $I \subseteq S$ is homogeneous such that $\text{depth}(S/I) \geq 3$, then

$$H_i^{n-2}(S) = H_i^{n-1}(S) = H_i^n(S) = 0$$

Sketch of the proof: After some manipulations we see, thanks to a result of **Ogus**, that we need to show that $H_{\text{DR},m}^2(\text{Spec}(S/I)) = 0$. In our setting, from a work of **Harthorne** this is equivalent to prove:

$$H^1(X, \mathbb{C}) = 0,$$

where $X = \text{Proj}(S/I)$ and we are assuming $k = \mathbb{C}$. The universal coefficient theorem and the exponential sequence make the job:

$$H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathcal{O}_X) \cong H_m^2(S/I)_0 = 0.$$

□

Set-theoretically Cohen-Macaulayness

Corollary: Let $X \subseteq \mathbb{P}^{n-1}$ be a smooth surface with $H^1(X, \mathcal{O}_X) \neq 0$ over a field of characteristic 0. Then for any graded ideal $I \subseteq S$ defining X set-theoretically S/I is not Cohen-Macaulay.

Proof. One can show that $H_i^{n-2}(S)$ (which does not depend on the radical of I !) is not zero. \square

Question: Are there analog examples for connected curves? Can you find a graded ideal $I \subseteq \mathbb{C}[a, b, c, d]$ defining set-theoretically $X = \{[s^4, s^3t, st^3, t^4] : [s, t] \in \mathbb{P}^1\} \subseteq \mathbb{P}^3$ such that $\mathbb{C}[a, b, c, d]/I$ is Cohen-Macaulay?

THANK YOU !!!