Singularities of ASL's

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Motivations

In the 80's Baclawski, De Concini, Eisenbud and Procesi introduced the notion of **Algebra with Straightening Laws (ASL)**.

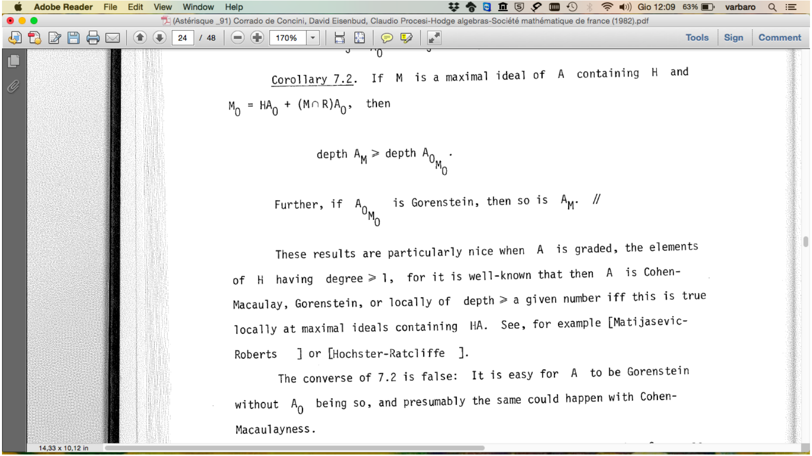
The ASL notion arose as an axiomatization of the combinatorial structure observed by many authors in classical algebraic varieties.

E.g. coordinate rings of Grassmannians, Shubert varieties and various kinds of rings defined by determinant equations are ASL.

Any ASL A has a discrete counterpart A_D (of the same dimension): A and A_D have a lot in common, and being defined by a square-free monomial ideal A_D is much simpler to study than A...

Motivations

From the '82 paper by De Concini, Eisenbud and Procesi:



Corollary 7.2. If M is a maximal ideal of A containing H and $M_0 = HA_0 + (M \cap R)A_0$, then

$$\text{depth } A_M \geqslant \text{depth } A_{0_{M_0}} .$$

Further, if $A_{0_{M_0}}$ is Gorenstein, then so is A_M. //

These results are particularly nice when A is graded, the elements of H having degree $\geqslant 1$, for it is well-known that then A is Cohen-Macaulay, Gorenstein, or locally of depth $\geqslant$ a given number iff this is true locally at maximal ideals containing HA. See, for example [Matijasevic-Roberts] or [Hochster-Ratcliffe].

The converse of 7.2 is false: It is easy for A to be Gorenstein without A_0 being so, and presumably the same could happen with Cohen-Macaulayness.

Motivations

In our notation the result of DEP says:

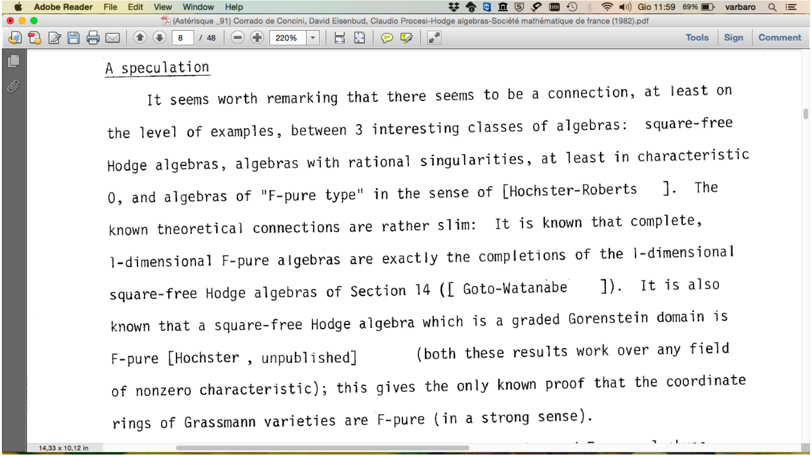
$$\operatorname{depth} A \geq \operatorname{depth} A_D$$

A recent result of Conca and myself implies that the equality holds; in particular, A is CM $\iff$ A_D is CM. Our result is more precise: it also implies that A is CM on the punctured spectrum if and only if A_D is so. In particular,

$$A \text{ isolated singularity } \implies A_D \text{ Buchbaum.}$$

This lead Constantinescu, De Negri and myself to study in a deeper way the singularities of an ASL...

As we learnt, these kind of speculations had already been done in the 80's. The text below is again from DEP:



A speculation

It seems worth remarking that there seems to be a connection, at least on the level of examples, between 3 interesting classes of algebras: square-free Hodge algebras, algebras with rational singularities, at least in characteristic 0, and algebras of "F-pure type" in the sense of [Hochster-Roberts]. The known theoretical connections are rather slim: It is known that complete, 1-dimensional F-pure algebras are exactly the completions of the 1-dimensional square-free Hodge algebras of Section 14 ([Goto-Watanabe]). It is also known that a square-free Hodge algebra which is a graded Gorenstein domain is F-pure [Hochster , unpublished] (both these results work over any field of nonzero characteristic); this gives the only known proof that the coordinate rings of Grassmann varieties are F-pure (in a strong sense).

Motivations

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The apparent connection of rational singularities and F-pure algebras was noted by Hochster.

As for the connection of Hodge algebras to algebras with rational singularities, one may remark first that all the main examples of square-free Hodge algebras given in Chapter VI, below, are known to have rational singularities at the ideals generated by H; of course, for a ring to have rational singularities it must be a Cohen-Macaulay domain, so presumably one should only look at square-free Hodge algebras which are domains and which are governed by "perfect" ideals of monomials (that is, A_0 should be Cohen-Macaulay; A_{HA} is then automatically Cohen-Macaulay too).

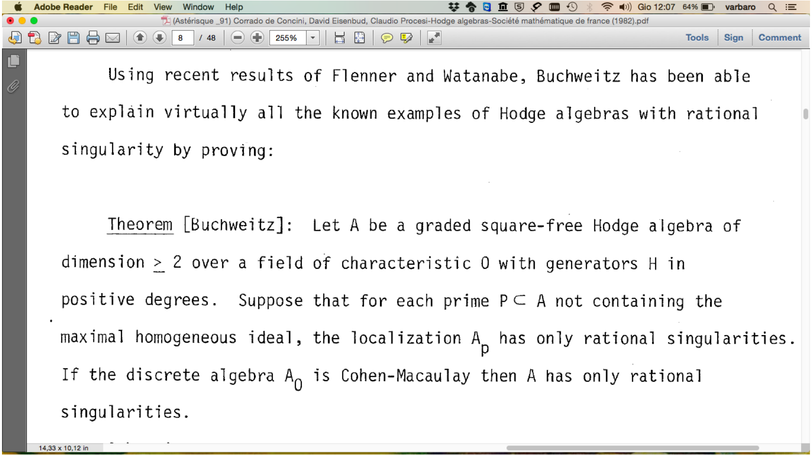
Eisenbud, in a '80 paper, even tried a conjecture:

more generally, a "Cohen-Macaulay") poset. At the moment, the most striking connection is that the known examples of wonderful algebras with straightening law are known to have rational singularities (when they are integral domains), and, in some cases, to be F-pure. Further, a recent result of Goto and Watanabe ([GW]) shows that in a reasonable setting, all complete 1-dimensional F-pure algebras are the completions of algebras with straightening laws (or, have complete straightening laws, in a reasonable sense). Further, recent work of Hochster and Huneke shows that an ASL is F-pure whenever it is a Gorenstein domain.

I suppose one reasonable conjecture is that every ASL A on a wonderful poset is F-pure (or of F-pure type in characteristic 0, in a strong sense) and is normal, with rational singularities, whenever A is a domain; but the situation is still very obscure.

Motivations

The results achieved with Constantinescu and De Negri go in the direction of confirming the feelings of De Concini, Eisenbud and Procesi. In the same direction was also the following unpublished result of Buchweitz (from DEP):



Using recent results of Flenner and Watanabe, Buchweitz has been able to explain virtually all the known examples of Hodge algebras with rational singularity by proving:

Theorem [Buchweitz]: Let A be a graded square-free Hodge algebra of dimension ≥ 2 over a field of characteristic 0 with generators H in positive degrees. Suppose that for each prime $P \subset A$ not containing the maximal homogeneous ideal, the localization A_P has only rational singularities. If the discrete algebra A_0 is Cohen-Macaulay then A has only rational singularities.

Definition and basic properties

- $A = \oplus_{i\in\mathbb{N}} A_i$ graded algebra with $A_0 = K$ a field;
- $(H, \prec)$ finite poset and $H \hookrightarrow \cup_{i>0} A_i$ an injective function;
- Given $h_1 \preceq h_2 \preceq \cdots \preceq h_s$ in H the corresponding product in A $a = h_1 \cdots h_s$ is called *standard monomial*. We denote $\min(a) = h_1$.

One says that A is an *Algebra with Straightening Laws on H* if:

1. The elements of H generate A as a K-algebra.
2. The standard monomials are K-linearly independent.
3. For every pair h_1, h_2 of incomparable elements of H there is a relation (called the straightening law)

$$h_1 h_2 = \sum_j \lambda_j a_j$$

where $\lambda_j \in K$ and a_j are standard monomials such that $\min(a_j) \prec h_1$ and $\min(a_j) \prec h_2$ for all j.

The three axioms above imply that the standard monomials form a basis of A over K and that the straightening laws define A as a quotient of the polynomial ring $K[H] = K[h : h \in H]$:

$$A = K[H]/I \quad \text{with} \quad I = (h_1 h_2 - \sum_j \lambda_j a_j : h_1 \not\prec h_2 \not\prec h_1).$$

The ideal $J = (h_1 h_2 : h_1 \not\prec h_2 \not\prec h_1)$ of $K[H]$ defines a quotient $A_D = K[H]/J$, *called the discrete ASL associated to* H. As it turns out, A_D is the special fiber of a flat family with general fiber A. This can be seen by observing that with respect to (weighted) degrevlex extending $\prec$ one has $J = \text{in}(I)$. Indeed, Conca observed that ASL's can also be defined via Gröbner degenerations.

F-purity of ASL's

It is easy to see that, if K has positive characteristic, a discrete ASL is F-pure. Unfortunately, F-purity does not deform (1999, Singh) so the F-purity of an ASL cannot be proved by a deformation argument. In fact Ohtani in 2013 gave an example that gives rise to an ASL which is not F-pure in characteristic 2. In this example, the underlying poset is not wonderful though... On the other hand, by a result of Ma and Quy of 2017:

F-injectivity of ASL's

If A is an ASL over a field of positive characteristic, it is F-injective.

Rationality of ASL's

Concerning the rationality of an ASL over a field of characteristic 0, in 1985 Hibi and Watanabe classified all Gorenstein ASL domains of dimension 3: in their classification, appears an example of ASL domain over a wonderful poset that is not normal (and so not a rational singularity).

It still makes sense to wonder if, assuming that A is a domain with rational singularities on the punctured spectrum, being an ASL implies that A itself is a rational singularity (and so normal and Cohen-Macaulay)

Our results

We could prove the following:

Constantinescu, De Negri, _ , 2019

Let A be an ASL domain which is CM on the punctured spectrum. Then the underlying poset is CM and contractible (in particular A is CM with negative a-invariant).

So the CM assumption of A_D in Buchweitz' result can be removed:

Corollary 0

Let A be an ASL domain of characteristic 0 with rational sing's on the punctured spectrum. Then A is a rational singularity.

Being an ASL F-injective, we also get the characteristic p analog:

Corollary p

Let A be an ASL domain of positive characteristic with F-rational sing's on the punctured spectrum. Then A is F-rational.

More general Gröbner deformations

Our proof works more generally, namely for all homogeneous ideals I in a polynomial ring S such that $\text{in}(I)$ is square-free for a degrevlex monomial order.

One ingredient of the proof is the following result solving a conjecture by Herzog, which holds for any monomial order:

Conca, _ , 2018

Let $I \subset S$ be a homogeneous ideal such that $\text{in}(I)$ is square-free. Then $\dim_K H^i_{\mathfrak{m}}(S/I)_j = \dim_K H^i_{\mathfrak{m}}(S/\text{in}(I))_j \ \forall \ i, j \in \mathbb{Z}$.

More general Gröbner deformations

Unfortunately the techniques we used to prove the ASL result cannot work for monomial orders different from degrevlex:

Example

Let $f = XYZ + Y^3 + Z^3 \in S = K[X, Y, Z]$. Then $S/(f)$ is a CM domain and, for lex extending $X > Y > Z$, $\text{in}(f) = (XYZ)$. However $S/(f)$ has not negative a-invariant.

It is also possible to produce a homogeneous ideal I of S such that S/I is a nonCM domain but CM in the punctured spectrum and with $\text{in}(I)$ square-free. So CM singularities on the punctured spectrum is not enough in general...

Gröbner degenerations of smooth projective varieties

If f is a homogeneous polynomial of a standard graded polynomial ring S in n variables, then $S/(f)$ has negative a-invariant if and only if $\deg(f) < n$. It is simple to check that, if $\mathrm{in}(f)$ is square-free for some monomial order and f defines a smooth projective variety, then $\deg(f) < n$. This suggested us the following:

Conjecture (Constantinescu, De Negri, _ , 2019)

Let I be a homogeneous prime ideal of a standard graded polynomial ring S such that $\mathrm{Proj}\, S/I$ is smooth. If $\mathrm{in}(I)$ is square-free for some monomial order, then S/I is Cohen-Macaulay and has negative a-invariant.

To our knowledge the above conjecture is open even for curves: By a result obtained in 1995 by Kalkbrener and Sturmfels, if $\operatorname{Proj} S/I$ is an irreducible curve then $\sqrt{\operatorname{in}(I)}$ is the Stanley-Reisner ideal of a connected graph. So the Cohen-Macaulayness of $S/\operatorname{in}(I)$ is already guaranteed if $\operatorname{in}(I)$ is square-free.

The fact that S/I has negative invariant, if $C = \operatorname{Proj} S/I$ is a smooth projective curve, is equivalent to the fact that C has genus 0, which under the assumption that $\operatorname{in}(I)$ is square-free is in turn equivalent to the fact that the above graph is a tree...

Fixed a monomial order, for the moment we can show the following:

Constantinescu, De Negri, _ , 2019

Let Δ be a graph without leaves. Then there is no homogeneous ideal I of S defining a smooth curve such that $\mathrm{in}(I) = I_\Delta$.

In particular, cycles do not admit a "Gröbner smoothing"...

THANK YOU !

Some references:

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4. D. Eisenbud, *Introduction to algebras with straightening laws*, Lect. Not. in Pure and Appl. Math. 55, 243–268, 1980.