Defining equations of algebraic sets

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How many polynomials are needed to define an algebraic set?

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This, however, is far to be true .....

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we claim that $X_1 = \mathcal{Z}(a,b,c) = \mathcal{Z}(f,g)$. Obviously $\mathcal{Z}(a,b,c)$ is contained in $\mathcal{Z}(f,g)$, for the other inclusion we have to prove that any point $P \in \mathcal{Z}(f,g)$ satisfies $b(P) = c(P) = 0$.

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$$c(P) = P_0P_3 - P_1P_2 = P_1^2P_3 - P_1 = P_1 - P_1 = 0.$$

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$$\mathrm{ara}_{\mathbb{P}^n} X = \min\{r \mid \exists\, f_1, \ldots, f_r \in S : X = \mathcal{Z}(f_1, \ldots, f_r)\}.$$

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We can therefore conclude that, in Example I:

$$\operatorname{ara}_{\mathbb{P}^3} X_1 = 2 = \operatorname{codim}_{\mathbb{P}^3} X_1.$$

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If $Y \subseteq \mathbb{P}^n$ is an irreducible d-dimensional algebraic set and $X \subseteq \mathbb{P}^n$ is the zero-locus of $< d$ polynomials, then $X \cap Y$ must be connected.

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It therefore makes sense to name $X \subseteq \mathbb{P}^n$ a **set-theoretic complete intersection** if $\operatorname{ara}_{\mathbb{P}^n} X = \operatorname{codim}_{\mathbb{P}^n} X$.

Example II

Let $X_2 = \{[s,t,0,0] : [s,t] \in \mathbb{P}^1\} \cup \{[0,0,s,t] : [s,t] \in \mathbb{P}^1\} \subseteq \mathbb{P}^3$.

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$$\mathcal{I}(X_2) = (x_2, x_3) \cap (x_0, x_1) = (x_0x_2,\ x_0x_3,\ x_1x_2,\ x_1x_3) \subseteq S.$$

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So $X_2 = \mathcal{Z}(I)$, and $\operatorname{ara}_{\mathbb{P}^3} X_2 = 3$.

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How to produce other lower bounds?

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$$\mathbb{P}^n \setminus X = U(f_1) \cup \ldots \cup U(f_r).$$

Therefore, by means of the Čech cohomology, $H^i(\mathbb{P}^n \setminus X, \mathcal{F}) = 0$ for any $i \geq r$ and each quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$.

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For what said above, for any algebraic set $X \subseteq \mathbb{P}^n$ we have:

$$\mathrm{ara}_{\mathbb{P}^n} X \geq \mathrm{cd}(\mathbb{P}^n \setminus X) + 1.$$

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How can we do in positive characteristic?

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However, in 1990 Bruns and Schwänzl managed to prove that $\mathrm{ara}_{\mathbb{P}^7} X_3 = 5$ also in positive characteristic .....

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Because $\text{écd}(U) \leq n$ whenever U is affine, Mayer-Vietoris yields:

$$\text{ara}_{\mathbb{P}^n} X \geq \text{écd}(U) - n + 1 \quad \text{for any algebraic set } X \subseteq \mathbb{P}^n.$$

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So far we have learnt that, given an algebraic set $X \subseteq \mathbb{P}^n$:

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Is $\text{écd}(\mathbb{P}^n \setminus X) \geq \text{codim}_{\mathbb{P}^n} X + n - 1 \;\; \forall$ algebraic set $X \subseteq \mathbb{P}^n$?

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The above question was originally stated by Hartshorne in 1979, when he proved that the above rational curve is a set-theoretic complete intersection in positive characteristic .....

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