ON THE DUAL GRAPHS OF COMPLETE INTERSECTIONS

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its corresponding homogeneous ideal.
- By the Nullstellensatz, we have $\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}$.

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An algebraic variety $X \subseteq \mathbb{P}^n$ of codimension c is a complete intersection (CI) if $\mathcal{I}(X)$ is generated by c polynomials.

An algebraic variety $X \subseteq \mathbb{P}^n$ is called a subspace arrangement if it is the union of linear subspaces of $\mathbb{P}^n$.

Line arrangements in $\mathbb{P}^3$

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We are going to inquire on the connectedness properties of $G(C)$ given global properties of C.

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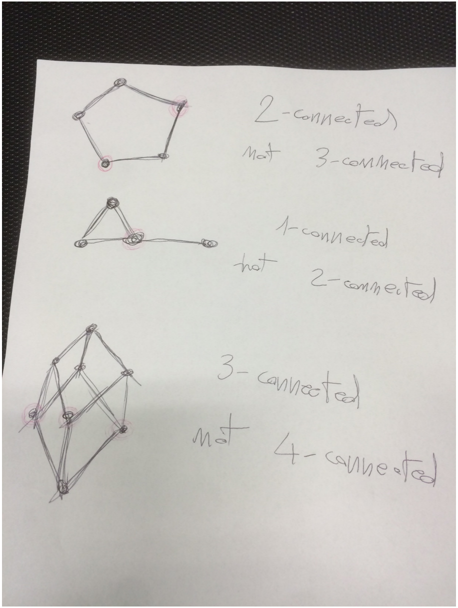
A graph G is r-connected if it has at least $r + 1$ vertices and removing $< r$ vertices yields a connected graph. In particular:

- G is connected $\Leftrightarrow$ G is 1-connected;
- G is $(r + 1)$-connected $\Rightarrow$ G is r-connected.

Examples of r-connectivity



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THEOREM (Benedetti, -): In the above situation, if $\deg(f) = d$ and $\deg(g) = e$, then $G(C)$ is $(d + e - 2)$-connected.

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For example, if the ideal of definition of C is defined by 2 cubics, then $G(C)$ will be 4-connected.

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In this case $f = \ell_{11} \cdots \ell_{1d}$ and $g = \ell_{21} \cdots \ell_{2e}$ will do the job. If the ℓ_{ij}'s are general enough, precisely if each four of them are linearly independent, then it turns out that the dual graph of C is not $(d + e - 1)$-connected.

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EXAMPLE: If $C = \{[s^3, s^2t, st^2, t^3] : [s,t] \in \mathbb{P}^1\} \subseteq \mathbb{P}^3$, then C is an algebraic curve and

$$\mathcal{I}(C) = (x_0x_2 - x_1^2,\ x_1x_3 - x_2^2,\ x_0x_3 - x_1x_2) \subseteq S = K[x_0, x_1, x_2, x_3].$$

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In particular, since $\mathcal{I}(C)$ needs 3 generators, C is not a CI. We will see soon that, however, C is a SCI .....

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This shows that $b(P) = 0$, and so that

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In over 40 years nobody could find 2 algebraic equations defining C, but how to show that they do not exist???

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Notice that C is connected if and only if the dual graph $G(C)$ is connected. Therefore the above result implies that is plenty of line arrangements which are SCI without being a CI .....

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By construction, the dual graph $G(C)$ is connected, so C is a set-theoretic complete intersection. However, it is not a complete intersection whenever $N \geq 3$, because $G(C)$ is not 2-connected ...

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If X is a complete intersection, then it is arithmetically Gorenstein. Furthermore, the Castelnuovo-Mumford regularity of $S/\mathcal{I}(X)$ is

$$\sum_{i=1}^{c} \deg(f_i) - c$$

where $\mathcal{I}(X) = (f_1, \ldots, f_c)$ and $c = \operatorname{codim}_{\mathbb{P}^n} X$.

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For example, if X is a linear space then its degree is 1.

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This recovers the result for subspace arrangements, since each irreducible component, being a linear space, has degree 1.

THANK YOU !!

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