Dual graphs and the Castelnuovo-Mumford regularity of subspace arrangements

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Dual graphs of simplicial complexes

If Δ is a simplicial complex on n vertices, its **dual graph** is the simple graph $G(\Delta)$ defined like follows:

- the vertices of $G(\Delta)$ correspond to the facets of Δ;
- $\{\sigma, \tau\}$ is an edge if and only if $\dim \sigma \cap \tau = \dim \Delta - 1$.

It is interesting to study the connectedness properties of $G(\Delta)$: of course $G(\Delta)$ in general is not even connected, but *the philosophy is that $G(\Delta)$ has good connectedness properties if Δ is a nice simplicial complex*.

Two ways to measure the connectedness of a simple graph G on a finite set of vertices V are provided by the following invariants:

- the **diameter** of G:

$$\text{diam } G := \max\{d(u,v) : u, v \text{ are vertices of } G\}$$

- the **connectivity** of G:

$$\kappa G := \min\{|A| : A \subset V,\ G_{V \setminus A} \text{ is disconnected or a single point}\}$$

Two typical examples of this philosophy are:

Balinski's theorem

If Δ is the boundary of a simplicial d-polytope, then $\kappa G(\Delta) = d$.

Hirsch's conjecture

If Δ is the boundary of a simplicial d-polytope on n vertices, then

$$\operatorname{diam} G(\Delta) \leq n - d.$$

The conjecture of Hirsch has been disproved in 2012 by Santos, however in general $n - d$ is a quite natural bound to expect for the diameter ...

For example is easy to see that normal simplicial complexes in low codimension have small diameter: Δ is **normal** if $G(\mathrm{lk}_\Delta \sigma)$ is connected for all $\sigma \in \Delta$. Clearly a boundary of a polytope is normal; more generally any triangulation of the sphere is normal, and even a Cohen-Macaulay simplicial complex is normal.

Let us see how normality translates for the **Stanley-Reisner ideal** of Δ,

$$I_\Delta := (x_{i_1} \cdots x_{i_k} : \{i_1, \ldots, i_k\} \notin \Delta) \subseteq K[x_1, \ldots, x_n].$$

We have that $I_\Delta = \bigcap_{\sigma \in \mathcal{F}(\Delta)} (x_i : i \notin \sigma)$ and the following are equivalent:

- Δ is normal.
- for any two minimal primes $\mathfrak{p}$ and $\mathfrak{p}'$ of I_Δ, there exist a sequence of minimal primes of I_Δ, $\mathfrak{p} = \mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_r = \mathfrak{p}'$ such that $\mathrm{height}(\mathfrak{p}_i + \mathfrak{p}_{i+1}) = \mathrm{height}(I_\Delta) + 1$ and $\mathfrak{p}_i \subseteq \mathfrak{p} + \mathfrak{p}'$.

Dual graphs

So to give a $(d-1)$-dimensional *normal* simplicial complex Δ on $[n]$ is like to give a collection $\mathcal{A}$ of subsets of $[n]$ of cardinality $h := n - d$ for which $\forall\ A, A' \in \mathcal{A}\ \exists\ A = A_0, A_1, \ldots, A_k = A'$ s.t.:

(i) $|A_i \cap A_{i-1}| = h - 1\ \forall\ i = 1, \ldots, k$.

(ii) $A_i \subseteq A \cup A'\ \forall\ i = 0, \ldots, k$.

Let us call such a collection $\mathcal{A}$ *locally connected* and write $G(\mathcal{A})$ for $G(\langle\mathcal{A}\rangle)$. If $h = 2$, obviously $\operatorname{diam} G(\mathcal{A}) \leq 2$. If $h = 3$, it is straightforward to check that $\operatorname{diam} G(\mathcal{A}) \leq 3$. However for larger h the analog statement is false, and it is not even known whether there is a polynomial $f(x) \in \mathbb{Z}[x]$ such that $\operatorname{diam} G(\mathcal{A}) \leq f(h)$.

This is the so-called **polynomial Hirsch conjecture** ...

Dual graphs

Adiprasito-Benedetti (2014)

If $\mathcal{A}$ is a locally connected collection of h-subsets of $[n]$ such that *any minimal vertex cover has cardinality* ≤ 2, then diam $G(\mathcal{A}) \leq h$.

In 2012 Bruno spoke of the above result within a conference in Genoa, and we began to wonder on what could be said on dual graphs of projective varieties ...

Dual graphs of projective varieties

Given a projective variety $X \subseteq \mathbb{P}^n$, if $X_1, \ldots, X_s$ are its irreducible components, we form the **dual graph** $G(X)$ as follows:

- The vertex set of $G(X)$ is $\{1, \ldots, s\}$.
- Two vertices $i \neq j$ are connected by an edge if and only if:

$$\dim(X_i \cap X_j) = \dim(X) - 1.$$

NOTE: If X is a projective curve, then $\{i, j\}$ is an edge if and only if $X_i \cap X_j \neq \emptyset$ (the empty set has dimension -1). If $\dim(X) > 1$, by intersecting $X \subseteq \mathbb{P}^n$ with a generic hyperplane, we get a projective variety in $\mathbb{P}^{n-1}$ of dimension one less, and same dual graph! Iterating this trick we can often reduce questions to curves.

Dual graphs of projective varieties

If Δ is a simplicial complex on $\{0, \ldots, n\}$, it is straightforward to check that $G(\Delta) = G(X)$, where $X = \mathcal{V}(I_\Delta) \subseteq \mathbb{P}^n$.

Today I want to discuss dual graphs of subspace arrangements, a particular case of projective varieties. A projective variety $X \subseteq \mathbb{P}^n$ is a **subspace arrangement** if its irreducible components are linear spaces. Notice that varieties of the type $\mathcal{V}(I_\Delta)$ are particular subspace arrangements. **How particular are they?**

The answer is: "very particular". The example below should be convincing: Let us take a codimension 2 simplicial complex Δ. By taking $\dim \Delta - 1$ general hyperplane sections, we obtain a line arrangement $X_\Delta \subseteq \mathbb{P}^3$ such that $G(X_\Delta) = G(\Delta)$.

(i) The normality of Δ corresponds to the fact that X is locally connected, which in this case is equivalent to say that X is connected. Of course there are connected line arrangements such that the dual graph has diameter as large as we want, while $\text{diam } G(X_\Delta) \leq 2$ in this case.

(ii) More interestingly, Δ being the boundary of a polytope corresponds to X being a complete intersection (i.e. $\mathcal{I}_X$ is generated by 2 polynomials). It is possible to produce a line arrangement $X \subseteq \mathbb{P}^3$ which is a complete intersection but $\text{diam } G(X) = 3$ ...

The Schläfli's double-six

Let G be the bipartite graph on $\{a_1, \ldots, a_6\} \cup \{b_1, \ldots, b_6\}$ such that $\{a_i, b_j\}$ is an edge if and only if $i \neq j$. Then diam $G = 3$.

The so-called *Schläfli's double-six* is a line arrangement $X \subseteq \mathbb{P}^3$ such that $G(X) = G$. It consists in 12 of the famous 27 lines on a smooth cubic $Y \subseteq \mathbb{P}^3$. Notice that the intersection points of X are 30, and the vector space of quartics of $\mathbb{P}^3$ has dimension 35. Therefore there is a quartic $Z \subseteq \mathbb{P}^3$ passing through these 30 points. Furthermore, by picking other 4 points outside of Y and not co-planar, one can also choose Z not containing Y. So $Y \cap Z$ is a complete intersection containing X, and since $3 \cdot 4 = 12$ we have $X = Y \cap Z$.

The Schläfli's double-six

In conclusion, the Schläfli's double-six is a line arrangement $X \subseteq \mathbb{P}^3$ that is a complete intersection of a cubic and a quartic but $\operatorname{diam} G(X) = 3$. It would be interesting to find examples of (d, e)-complete intersection line arrangements $X \subseteq \mathbb{P}^3$ whose dual graph has diameter arbitrarily large.

One complication is that the general surface of degree k in $\mathbb{P}^3$ contains no lines for $k \geq 4$, and $\min\{d, e\} \geq \operatorname{diam} G(X)$ by the following result ...

The connectivity number

Benedetti-V. (2015)

If $X \subseteq \mathbb{P}^3$ is a (d,e)-complete intersection line arrangement, then

$$\kappa G(X) \geq d + e - 2.$$

With Bruno we proved a more general statement, involving the Castelnuovo-Mumford regularity of X. I will describe it in the next slides, along with a *new* more precise result.

For the moment notice that, if $X \subseteq \mathbb{P}^3$ is a (d,e)-complete intersection line arrangement, then $G(X)$ has $d \cdot e$ vertices, so the result above and a classical theorem of Menger says that

$$\operatorname{diam} G(X) \leq \min\{d, e\}.$$

The Castelnuovo-Mumford regularity

Let $S = K[x_0, \ldots, x_n]$, $I \subseteq S$ a homogeneous ideal, and $R = S/I$. The **Hilbert function** $\mathrm{HF}_R : \mathbb{Z} \to \mathbb{Z}$, which maps d to $\dim_K R_d$, is eventually a polynomial HP_R, called the **Hilbert polynomial**.

If R is Cohen-Macaulay, its **Castelnuovo-Mumford regularity** is:

$$\operatorname{reg} R := \max\{d \in \mathbb{Z} : \mathrm{HF}_R(d) \neq \mathrm{HP}_R(d)\} + \dim R.$$

Reisner's criterion

For a simplicial complex Δ, the following are equivalent:

- $K[\Delta]$ is Cohen-Macaulay.
- $\widetilde{H}_i(\operatorname{lk}_\Delta \sigma; K) = 0$ for all $\sigma \in \Delta$ and $i < \dim \operatorname{lk}_\Delta \sigma$.

Gorenstein rings

Before stating the main result it's good to have an intuition of what a **Gorenstein ring** R is: it is a ring which is Cohen-Macaulay and has an extra feature, namely it satisfies the Poincaré duality once gone modulo a system of parameters.

A criterion for Gorenstein SR rings (Stanley)

For a simplicial complex Δ, the following are equivalent:

- $K[\Delta]$ is Gorenstein.
- Δ is the join of a homology sphere with a simplex.

Another class of Gorenstein rings is formed by complete intersections, namely rings of the form $R = S/I$ where I is generated by homogeneous polynomials $f_1, \ldots, f_c$, where c is the codimension of $\mathcal{V}(I) \subseteq \mathbb{P}^n$. In this case $\operatorname{reg} R = \sum_{i=1}^{c} \deg(f_i) - c$, while in the case of SR Gorenstein rings $\operatorname{reg} K[\Delta] - 1$ is the dimension of the homology sphere above. In general may be difficult to compute the regularity of a Gorenstein ring ...

Connectivity number VS Castelnuovo-Mumford regularity

Benedetti-V. (2015)

If $X \subseteq \mathbb{P}^n$ is a subspace arrangement and $R = S/\mathcal{I}_X$ is Gorenstein, then $\kappa G(X) \geq \operatorname{reg} R$.

As a very special case, this recovers Balinski's theorem. In general, however, $\kappa G(X)$ may be much bigger than $\operatorname{reg} R$ ...

Benedetti-Di Marca-V. (2016)

If $X = \bigcup_{i=1}^{s} V_i \subseteq \mathbb{P}^n$ is a subspace arrangement and $R = S/\mathcal{I}_X$ is Gorenstein, and $V_1 \cap V_i \cap V_j$ has codimension at least 2 in X for all $i < j$, then

$$\kappa G(X) = \operatorname{reg} R.$$

If moreover $V_i \cap V_j \cap V_k$ has codimension at least 2 in X for all $i < j < k$, then $G(X)$ is a $\operatorname{reg} R$-regular graph.