BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 45(4), 767-775

ISSN: 1678-7544 (Print) / 1678-7714 (Online)

The F-pure threshold of a determinantal ideal

Lance Edward Miller, Anurag K. Singh*
and Matteo Varbaro

— Dedicated to Professor Steven Kleiman and Professor Aron Simis on the occasion of their 70th birthdays.

Abstract. The F-pure threshold is a numerical invariant of prime characteristic singularities, that constitutes an analogue of the log canonical thresholds in characteristic zero. We compute the F-pure thresholds of determinantal ideals, i.e., of ideals generated by the minors of a generic matrix.

Keywords: F-pure threshold, log canonical threshold, determinantal ideals.

Mathematical subject classification: Primary: 13A35; Secondary: 13C40, 13A50.

1 Introduction

Consider the ring of polynomials in a matrix of indeterminates X, with coefficients in a field of prime characteristic. We compute the F-pure thresholds of determinantal ideals, i.e., of ideals generated by the minors of X of a fixed size.

The notion of *F-pure thresholds* is due to Takagi and Watanabe [18], see also Mustaţă, Takagi, and Watanabe [17]. These are positive characteristic invariants of singularities, analogous to log canonical thresholds in characteristic zero. While the definition exists in greater generality – see the above papers – the following is adequate for our purpose:

Definition 1.1. *Let R be a polynomial ring over a field of characteristic $p > 0$, with the homogeneous maximal ideal denoted by $\mathfrak{m}$. For a homogeneous proper ideal I, and integer $q = p^e$, set*

$$\nu_I(q) = \max\left\{r \in \mathbb{N} \mid I^r \not\subseteq \mathfrak{m}^{[q]}\right\},$$

Received 10 December 2012.
*Corresponding author.
L.E.M. was supported by VIGRE grant 0602219, and A.K.S. by NSF grant DMS 1162585. A.K.S. and M.V. were also supported by NSF grant 0932078000 while in residence at MSRI.

where $\mathfrak{m}^{[q]} = (a^q \mid a \in \mathfrak{m})$. *If* I *is generated by* N *elements, it is readily seen that* $0 \leqslant \nu_I(q) \leqslant N(q-1)$. *Moreover, if* $f \in I^r \setminus \mathfrak{m}^{[q]}$, *then* $f^p \in I^{pr} \setminus \mathfrak{m}^{[pq]}$. *Thus,*

$$\nu_I(pq) \geqslant p\nu_I(q)\,.$$

It follows that $\left\{\nu_I(p^e)/p^e\right\}_{e \geqslant 1}$ *is a bounded monotone sequence; its limit is the* F-pure threshold of I, *denoted* $\mathrm{fpt}(I)$.

The F-pure threshold is known to be rational in a number of cases, see, for example, [2, 3, 4, 9, 16]. The theory of F-pure thresholds is motivated by connections to log canonical thresholds; for simplicity, and to conform to the above context, let I be a homogeneous ideal in a polynomial ring over the field of rational numbers. Using "I modulo p" to denote the corresponding characteristic p model, one has the inequality

$$\mathrm{fpt}(I \text{ modulo } p) \leqslant \mathrm{lct}(I) \qquad \text{for all } \ p \gg 0\,,$$

where $\mathrm{lct}(I)$ denotes the log canonical threshold of I. Moreover,

$$\lim_{p \longrightarrow \infty} \mathrm{fpt}(I \text{ modulo } p) = \mathrm{lct}(I)\,. \tag{1.1.1}$$

These follow from work of Hara and Yoshida [10]; see [17, Theorems 3.3, 3.4].

The F-pure thresholds of defining ideals of Calabi-Yau hypersurfaces are computed in [1]. Hernández has computed F-pure thresholds for binomial hypersurfaces [11] and for diagonal hypersurfaces [12]. In the present paper, we perform the computation for determinantal ideals:

Theorem 1.2. *Fix positive integers* $t \leqslant m \leqslant n$, *and let* X *be an* $m \times n$ *matrix of indeterminates over a field* $\mathbb{F}$ *of prime characteristic. Let* R *be the polynomial ring* $\mathbb{F}[X]$, *and* I_t *the ideal generated by the size* t *minors of* X.

The F*-pure threshold of* I_t *is*

$$\min\left\{\frac{(m-k)(n-k)}{t-k} \mid k = 0, \ldots, t-1\right\}.$$

It follows that the F-pure threshold of a determinantal ideal is independent of the characteristic: for each prime characteristic, it agrees with the log canonical threshold of the corresponding characteristic zero determinantal ideal, as computed by Johnson [15, Theorem 6.1] or Docampo [8, Theorem 5.6] using log resolutions as in Vainsencher [19]. In view of (1.1.1), Theorem 1.2 recovers the calculation of the characteristic zero log canonical threshold.

2 The computations

The primary decomposition of powers of determinantal ideals, i.e., of the ideals I_t^s, was computed by DeConcini, Eisenbud, and Procesi [7] in the case of characteristic zero, and extended to the case of *non-exceptional* prime characteristic by Bruns and Vetter [6, Chapter 10]. By Bruns [5, Theorem 1.3], the intersection of the primary ideals arising in a primary decomposition of I_t^s in non-exceptional characteristics, yields, in all characteristics, the integral closure $\overline{I_t^s}$. We record this below in the form that is used later in the paper:

Theorem 2.1 (Bruns). *Let s be a positive integer, and let $\delta_1, \dots, \delta_h$ be minors of the matrix X. If*

$$h \leqslant s \quad \text{and} \quad \sum_i \deg \delta_i = ts,$$

then

$$\delta_1 \cdots \delta_h \ \in \ \overline{I_t^s}\,.$$

Proof. By [5, Theorem 1.3], the ideal $\overline{I_t^s}$ has a primary decomposition

$$\bigcap_{j=1}^{t} I_j^{((t-j+1)s)}\,.$$

Thus, it suffices to verify that

$$\delta_1 \cdots \delta_h \ \in \ I_j^{((t-j+1)s)}$$

for each j with $1 \leqslant j \leqslant t$. This follows from [6, Theorem 10.4]. □

We will also need:

Lemma 2.2. *Let k be the least integer in the interval $[0, t-1]$ such that*

$$\frac{(m-k)(n-k)}{t-k} \ \leqslant \ \frac{(m-k-1)(n-k-1)}{t-k-1};$$

interpreting a positive integer divided by zero as infinity, such a k indeed exists. Set

$$u \ = \ t(m+n-2k) - mn + k^2\,.$$

Then $t-k-u \geqslant 0$.

Moreover, if k is nonzero, then $t-k+u > 0$; if $k = 0$, then $t(m+n-1) \leqslant mn$.

Proof. Rearranging the inequality above, we have

$$t(m+n-2k-1) \leqslant mn-k^2-k\,,$$

which gives $t-k-u \geqslant 0$. If k is nonzero, then the minimality of k implies that

$$t(m+n-2k+1) > mn-k^2+k\,,$$

equivalently, that $t-k+u > 0$. If $k=0$, the assertion is readily verified. □

Notation 2.3. Let X be an $m \times n$ matrix of indeterminates. Following the notation in [6], for indices

$$1 \leqslant a_1 < \cdots < a_t \leqslant m \qquad \text{and} \qquad 1 \leqslant b_1 < \cdots < b_t \leqslant n\,,$$

we set $[a_1,\ldots,a_t \mid b_1,\ldots,b_t]$ to be the minor

$$\det\begin{pmatrix} x_{a_1b_1} & \ldots & x_{a_1b_t} \\ \vdots & & \vdots \\ x_{a_tb_1} & \ldots & x_{a_tb_t} \end{pmatrix}.$$

We use the lexicographical term order on $R=\mathbb{F}[X]$ with

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{m1} > \cdots > x_{mn}\,;$$

under this term order, the initial form of the minor displayed above is the product of the entries on the leading diagonal, i.e.,

$$\operatorname{in}\big([a_1,\ldots,a_t \mid b_1,\ldots,b_t]\big) = x_{a_1b_1}x_{a_2b_2}\cdots x_{a_tb_t}\,.$$

For an integer k with $0 \leqslant k \leqslant m$, we set Δ_k to be the product of minors:

$$\prod_{i=1}^{n-m+1}[1,\ldots,m \mid i,\ldots,i+m-1]$$

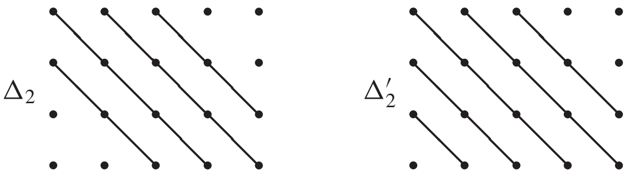
$$\times\prod_{j=2}^{m-k}[j,\ldots,m \mid 1,\ldots,m-j+1]\cdot[1,\ldots,m-j+1 \mid n-m+j,\ldots,n]\,.$$

If $k \geqslant 1$, we set Δ'_k to be

$$\Delta_k\cdot[m-k+1,\ldots,m \mid 1,\ldots,k]\,.$$

Notice that $\deg\Delta_k = mn-k^2-k$ and that $\deg\Delta'_k = mn-k^2$. The element Δ_k is a product of $m+n-2k-1$ minors and Δ'_k of $m+n-2k$ minors.

Example 2.4. We include an example to assist with the notation. In the case $m = 4$ and $n = 5$, the elements Δ_2 and Δ_2' are, respectively, the products of the minors determined by the leading diagonals displayed below:



The initial form of Δ_2' is the square-free monomial

$$x_{11}\,x_{12}\,x_{13}\,x_{21}\,x_{22}\,x_{23}\,x_{24}\,x_{31}\,x_{32}\,x_{33}\,x_{34}\,x_{35}\,x_{42}\,x_{43}\,x_{44}\,x_{45}\,.$$

For arbitrary m, n, the initial form of Δ_0 is the product of the mn indeterminates.

Proof of Theorem 1.2. We first show that for each k with $0 \leqslant k \leqslant t-1$, one has

$$\operatorname{fpt}(I_t) \;\leqslant\; \frac{(m-k)(n-k)}{t-k}\,.$$

Let δ_k and δ_t be minors of size k and t respectively. Theorem 2.1 implies that

$$\delta_k^{t-k-1}\delta_t \;\in\; \overline{I_{k+1}^{t-k}}\,,$$

and hence that $\delta_k^{t-k-1} I_t \;\subseteq\; \overline{I_{k+1}^{t-k}}$. By the Briançon-Skoda theorem, see, for example, [13, Theorem 5.4], there exists an integer N such that

$$\left(\delta_k^{t-k-1} I_t\right)^{N+l} \;\in\; I_{k+1}^{(t-k)l}$$

for each integer $l \geqslant 1$. Localizing at the prime ideal I_{k+1} of R, one has

$$I_t^{N+l} \;\subseteq\; I_{k+1}^{(t-k)l} R_{I_{k+1}} \qquad \text{for each } \; l \geqslant 1\,,$$

as the element δ_k is a unit in $R_{I_{k+1}}$. Since $R_{I_{k+1}}$ is a regular local ring of dimension $(m-k)(n-k)$, with maximal ideal $I_{k+1} R_{I_{k+1}}$, it follows that

$$I_t^{N+l} \;\subseteq\; I_{k+1}^{[q]} R_{I_{k+1}}$$

for positive integers l and $q = p^e$ satisfying

$$(t-k)l \;>\; (q-1)(m-k)(n-k)\,.$$

Returning to the polynomial ring R, the ideal I_{k+1} is the unique associated prime of $I_{k+1}^{[q]}$; this follows from the flatness of the Frobenius endomorphism, see for example, [14, Corollary 21.11]. Hence, in the ring R, we have

$$I_t^{N+l} \subseteq I_{k+1}^{[q]}$$

for all integers q, l satisfying the above inequality. This implies that

$$\nu_{I_t}(q) \leqslant N + \frac{(q-1)(m-k)(n-k)}{t-k}.$$

Dividing by q and passing to the limit, one obtains

$$\operatorname{fpt}(I_t) \leqslant \frac{(m-k)(n-k)}{t-k}.$$

Next, fix k and u be as in Lemma 2.2, and consider Δ_k and Δ'_k as in Notation 2.3; the latter is defined only in the case $k \geqslant 1$. Set

$$\Delta = \begin{cases} \Delta_0^t & \text{if} \quad k = 0, \\ \Delta_k^u \cdot (\Delta'_k)^{t-k-u} & \text{if} \quad k \geqslant 1 \text{ and } u \geqslant 0, \\ (\Delta'_k)^{t-k+u} \cdot \Delta_{k-1}^{-u} & \text{if} \quad k \geqslant 1 \text{ and } u < 0, \end{cases}$$

bearing in mind that $t - k - u \geqslant 0$ by Lemma 2.2.

We claim that Δ belongs to the integral closure of the ideal $I_t^{(m-k)(n-k)}$. This holds by Theorem 2.1, since, in each case,

$$\deg \Delta = t(m-k)(n-k),$$

and Δ is a product of at most $(m-k)(n-k)$ minors: if $k \geqslant 1$, then Δ is a product of exactly $(m-k)(n-k)$ minors, whereas if $k = 0$ then Δ is a product of $t(m+n-1)$ minors and, by Lemma 2.2, one has $t(m+n-1) \leqslant mn$.

Let $\mathfrak{m}$ be the homogeneous maximal ideal of R. For a positive integer s that is not necessarily a power of p, set

$$\mathfrak{m}^{[s]} = (x_{ij}^s \mid i = 1, \dots, m, \;\; j = 1, \dots, n).$$

Using the lexicographical term order from Notation 2.3, the initial forms $\operatorname{in}(\Delta_k)$ and $\operatorname{in}(\Delta'_k)$ are square-free monomials, and

$$\operatorname{in}(\Delta) = \begin{cases} \operatorname{in}(\Delta_0)^t & \text{if} \quad k = 0, \\ \operatorname{in}(\Delta_k)^u \cdot \operatorname{in}(\Delta'_k)^{t-k-u} & \text{if} \quad k \geqslant 1 \text{ and } u \geqslant 0, \\ \operatorname{in}(\Delta'_k)^{t-k+u} \cdot \operatorname{in}(\Delta_{k-1})^{-u} & \text{if} \quad k \geqslant 1 \text{ and } u < 0. \end{cases}$$

Thus, each variable x_{ij} occurs in the monomial $\mathrm{in}(\Delta)$ with exponent at most $t-k$. It follows that

$$\Delta \notin \mathfrak{m}^{[t-k+1]}.$$

As Δ belongs to the integral closure of $I_t^{(m-k)(n-k)}$, there exists a nonzero homogeneous polynomial $f \in R$ such that

$$f\Delta^l \in I_t^{(m-k)(n-k)l} \qquad \text{for all integers } l \geqslant 1\,.$$

But then

$$f\Delta^l \in I_t^{(m-k)(n-k)l} \setminus \mathfrak{m}^{[q]}$$

for all integers l with $\deg f + l(t-k) \leqslant q-1$. Hence,

$$\nu_{I_t}(q) \geqslant (m-k)(n-k)l \qquad \text{for all integers } l \text{ with } l \leqslant \frac{q-1-\deg f}{t-k}\,.$$

Thus,

$$\nu_{I_t}(q) \geqslant (m-k)(n-k)\left(\frac{q-1-\deg f}{t-k}-1\right),$$

and dividing by q and passing to the limit, one obtains

$$\mathrm{fpt}(I_t) \geqslant \frac{(m-k)(n-k)}{t-k},$$

which completes the proof. □

References

[1] B. Bhatt and A.K. Singh. *The F-pure threshold of a Calabi-Yau hypersurface*, preprint, `arXiv:1307.1171 [math.AC]`.

[2] M. Blickle, M. Mustaţă and K.E. Smith. *Discreteness and rationality of F-thresholds*. Michigan Math. J., **57** (2008), 43–61.

[3] M. Blickle, M. Mustaţă and K.E. Smith. *F-thresholds of hypersurfaces*. Trans. Amer. Math. Soc., **361** (2009), 6549–6565.

[4] M. Blickle, K. Schwede, S. Takagi and W. Zhang. *Discreteness and rationality of F-jumping numbers on singular varieties*. Math. Ann., **347** (2010), 917–949.

[5] W. Bruns. *Algebras defined by powers of determinantal ideals*. J. Algebra, **142** (1991), 150–163.

[6] W. Bruns and U. Vetter. *Determinantal rings*. Lecture Notes in Mathematics, **1327**, Springer-Verlag, Berlin, (1988).

[7] C. DeConcini, D. Eisenbud and C. Procesi. *Young diagrams and determinantal varieties*. Invent. Math., **56** (1980), 129–165.

[8] R. Docampo. *Arcs on determinantal varieties*. Trans. Amer. Math. Soc., **365** (2013), 2241–2269.

[9] N. Hara. *F-pure thresholds and F-jumping exponents in dimension two*, with an appendix by P. Monsky. Math. Res. Lett., **13** (2006), 747–760.

[10] N. Hara and K.-I. Yoshida. *A generalization of tight closure and multiplier ideals*. Trans. Amer. Math. Soc., **355** (2003), 3143–3174.

[11] D.J. Hernández. *F-pure thresholds of binomial hypersurfaces*. Proc. Amer. Math. Soc., **142** (2014), 2227–2242.

[12] D.J. Hernández. *F-invariants of diagonal hypersurfaces*. Proc. Amer. Math. Soc., to appear, `arXiv:1112.2425 [math.AC]`.

[13] M. Hochster and C. Huneke. *Tight closure, invariant theory, and the Briançon-Skoda theorem*. J. Amer. Math. Soc., **3** (1990), 31–116.

[14] S.B. Iyengar, G.J. Leuschke, A. Leykin, C. Miller, E. Miller, A.K. Singh and U. Walther. *Twenty-four hours of local cohomology*. Grad. Stud. Math., **87**, American Mathematical Society, Providence, RI, (2007).

[15] A.A. Johnson. *Multiplier ideals of determinantal ideals*. Ph.D. Thesis, University of Michigan, (2003).

[16] M. Katzman, G. Lyubeznik and W. Zhang. *On the discreteness and rationality of F-jumping coefficients*. J. Algebra, **322** (2009), 3238–3247.

[17] M. Mustaţă, S. Takagi and K.-i. Watanabe. *F-thresholds and Bernstein-Sato polynomials*. European Congress of Mathematics, 341–364, Eur. Math. Soc., Zürich, (2005).

[18] S. Takagi and K.-i. Watanabe. *On F-pure thresholds*. J. Algebra, **282** (2004), 278–297.

[19] I. Vainsencher. *Complete collineations and blowing up determinantal ideals*. Math. Ann., **267** (1984), 417–432.

Lance Edward Miller
Department of Mathematical Sciences
University of Arkansas
Fayetteville
AR 72701
USA

E-mail: lem016@uark.edu

Anurag K. Singh
Department of Mathematics
University of Utah
155 S 1400 E
Salt Lake City
UT 84112
USA

E-mail: singh@math.utah.edu

Matteo Varbaro
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
I-16146 Genova
ITALY

E-mail: varbaro@dima.unige.it