ARITHMETICAL RANK
OF CERTAIN SEGRE PRODUCTS

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Clearly $\mathrm{ara}(I) \leq \mathrm{ara}_h(I)$.

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After them Song, student of Singh, proved the same statement when $f = x_0^d + x_1^d + x_2^d$, $n = 2$, $m = 1$ ($(\mathrm{char}(K), d) = 1$).

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(2) $\mathrm{ara}_h(I) \geq N-1$ if $n=2,\ d(=\deg(f)) \geq 3\ \ (X$ smooth$)$

We define two sets of polynomials of R of degree d, $\mathcal{P}^n_d$ and $\mathcal{L}^n_d$:

$f \in \mathcal{P}^n_d \Leftrightarrow f = x_n^d + g$ with $g \in (x_0, \ldots, x_{n-2})$ (up to change of coordinates)

$f \in \mathcal{L}^n_d \Leftrightarrow f = x_n^d + g$ with $g \in (x_0, \ldots, x_{n-3})$ (up to change of coordinates)

(3) $\mathrm{ara}_h(I) \leq N-1$ for any n, $m=1$, $f \in \mathcal{P}^n_d$

$\mathcal{P}^n_d$ contains the general polynomial f of degree $d \leq 2n-1$.

$\mathcal{P}^n_d$ contains any polynomial $f = x_0^d + \ldots + x_n^d$

$\mathcal{P}^2_d$ has codimension $d-3$ in the Hilbert scheme of curves of degree d.

(4) $\mathrm{ara}_h(I) \leq N-2$ for any n, $m=1$, $f \in \mathcal{L}^n_d$

(5) $\mathrm{ara}_h(I) = N-2$ for $n=2$ and X smooth conic (any m, $\mathrm{char}(K) \neq 2$)

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So the polynomials generating I are $\binom{n+1}{2} \cdot \binom{m+1}{2} + \binom{m+d}{d}$

UPPER BOUNDS

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Look at the case when $Y = X \times \mathbb{P}^1 \subseteq \mathbb{P}^5$ where $X = \mathcal{V}_+(f)$ is a smooth conic of $\mathbb{P}^2$ (so f is a polynomial of degree 2 in $R = K[x_0, x_1, x_2]$)

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So $\mathrm{ara}_h(I) = 3$, and Y is a set-theoretic complete intersection of $\mathbb{P}^5$.

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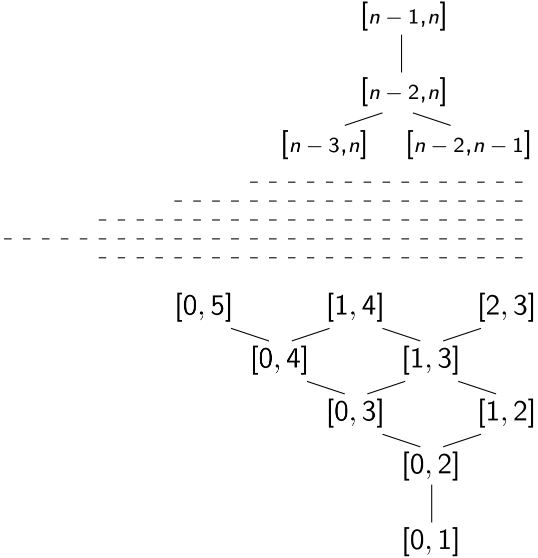

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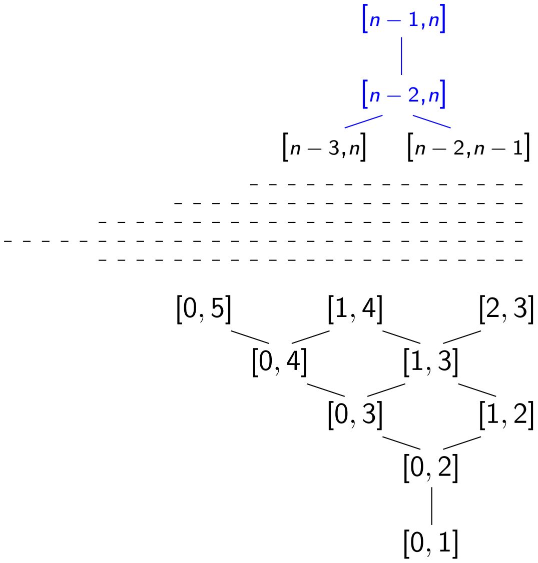

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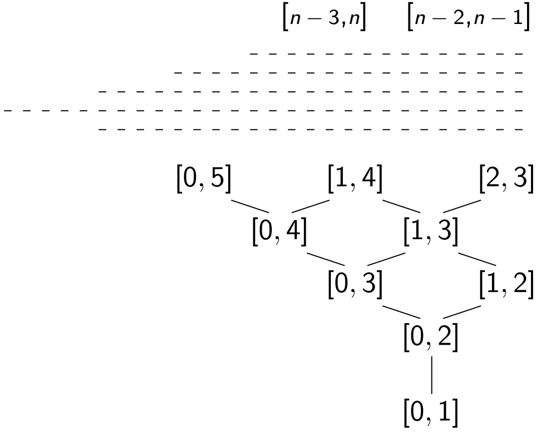

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It turns out that, if $n \geq 4$, the general X defined by a such f is smooth!

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If $n \geq 2$ is arbitrary, using very classical arguments of algebraic geometry one can show:

if $d \leq 2n - 3$ then every smooth hypersurface of degree d is of that kind.

If $d \leq 2n - 1$ then every *general* smooth hypersurface of degree d is so.

LOWER BOUNDS

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In this talk we will use local cohomology.

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We are interested in the case in which R is a polynomial ring.

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Question: $(R, \mathfrak{m})$ regular local ring, $I \subseteq R$. Is it true that

$$\mathrm{depth}(R/I) \geq 3 \Rightarrow \mathrm{cd}(R, I) \leq \dim(R) - 3 \text{ ?????}$$

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Q: Does the arithmetical rank depend only on n, m, d as well?