The nerve of a positively graded K-algebra

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Notation

- K is a field;
- R is a Noetherian positively graded K-algebra, i. e. $R = \bigoplus_{i=0}^{+\infty} R_i$ with $R_0 = K$;
- $\mathfrak{m} = \bigoplus_{i=1}^{+\infty} R_i$ is the maximal homogeneous ideal of R.

In the above setting, the minimal primes of R are homogeneous, and so contained in $\mathfrak{m}$.

Definition

If $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$, the **nerve** $\mathcal{N}(R)$ of R (also known as **Lyubeznik complex** of R) is the following simplicial complex:

- The vertex set of $\mathcal{N}(R)$ is $\{1, \ldots, s\}$;
- $\{i_1, \ldots, i_r\}$ is a face of $\mathcal{N}(R)$ $\iff$ $\sqrt{\mathfrak{p}_{i_1} + \ldots + \mathfrak{p}_{i_r}} \subsetneq \mathfrak{m}$.

The Stanley-Reisner case

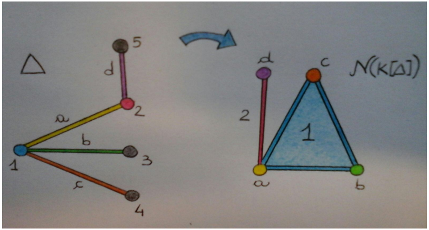
Example

Consider the simplicial complex on 5 vertices

$$\Delta = \langle \{1,2\}, \{1,3\}, \{1,4\}, \{2,5\} \rangle$$

and $R = K[\Delta]$. Then $R = \frac{K[X_1,X_2,X_3,X_4]}{(X_3,X_4,X_5)\cap(X_2,X_4,X_5)\cap(X_2,X_3,X_5)\cap(X_1,X_3,X_4)}$:

- $\mathfrak{p}_a = (x_3, x_4, x_5), \mathfrak{p}_b = (x_2, x_4, x_5), \mathfrak{p}_c = (x_2, x_3, x_5), \mathfrak{p}_d = (x_1, x_3, x_4)$, where $x_i = \overline{X_i}$, are the minimal primes of R.
- $\mathcal{N}(R)$ is the complex on $\{a, b, c, d\}$ with facets $\{a, b, c\}$ and $\{a, d\}$.



In general, given a simplicial complex Δ, the nerve of the ring $K[\Delta]$ is the nerve of Δ with respect to the covering of Δ given by its facets, i.e. $\mathcal{N}(K[\Delta])$ is the simplicial complex whose vertex set is $\mathcal{F}(\Delta)$, the set of facets of Δ, and such that

$$\mathcal{A} \subseteq \mathcal{F}(\Delta) \text{ is a face of } \mathcal{N}(K[\Delta]) \iff \bigcap_{\sigma \in \mathcal{A}} \sigma \neq \emptyset.$$

So, Borsuk's nerve lemma implies that

Δ **and** $\mathcal{N}(K[\Delta])$ **are homotopically equivalent**.

In particular, $\widetilde{H^i}(\mathcal{N}(K[\Delta]); K) = \widetilde{H^i}(\Delta; K) \quad \forall\ i \in \mathbb{N}$.

Classical results

In general (for any Noetherian positively graded K-algebra R):

Proposition (Hartshorne, 1962)

$\operatorname{Proj} R$ is connected if and only if $\mathcal{N}(R)$ is connected.

Theorem (Hartshorne, 1962)

If depth $R \geq 2$, then $\mathcal{N}(R)$ is connected.

Theorem (Bertini, ∼1920; Grothendieck, 1968)

If R is a domain of dimension ≥ 3, then $\mathcal{N}(R/xR)$ is connected whenever $x \in R_i$ with $i > 0$.

Classical results

Since $\mathcal{N}(R)$ is connected if and only if $\widetilde{H^0}(\mathcal{N}(R); K) = 0$, the results above can be rephrased like follows:

Theorem (Hartshorne, 1962)

If depth $R \geq 2$, then $\widetilde{H^0}(\mathcal{N}(R); K) = 0$.

Theorem (Bertini, ∼1920; Grothendieck, 1968)

If R is a domain of dimension ≥ 3, then $\widetilde{H^0}(\mathcal{N}(R/xR); K) = 0$ whenever $x \in R_i$ with $i > 0$.

More recent developements

Theorem (Katzman, Lyubeznik, Zhang, 2015)

If depth $R \geq 3$, then $\widetilde{H^0}(\mathcal{N}(R); K) = \widetilde{H^1}(\mathcal{N}(R); K) = 0$.

Question (Lyubeznik)

If R is a positively graded domain of dimension at least 4, is it true that $\widetilde{H^1}(\mathcal{N}(R/xR); K) = 0$ whenever $x \in R_i$ with $i > 0$?

Answer

No.

In this talk I will explain how to construct an example giving a negative answer; before, however, I want to tell you something about the inverse problem to the theorems seen so far...

Inverse problem

Question (Inverse problem)

Fixed $d \geq \delta$, is there a function attaching a d-dimensional positively graded K-algebra A_Δ to any $(\delta-1)$-dimensional simplicial complex Δ in such a way that properties (i), (ii), (iii), (iv) below are verified?

(i) $\mathcal{N}(A_\Delta) = \Delta$;

(ii) if $\mathrm{Min}(A_\Delta) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, $\sum_{i \in \sigma} \mathfrak{p}_i$ has height $\dim \sigma$ for all $\sigma \in \Delta$;

(iii) $\widetilde{H^0}(\Delta; K) = 0 \implies \operatorname{depth} A_\Delta \geq 2$;

(iv) $\widetilde{H^0}(\Delta; K) = \widetilde{H^1}(\Delta; K) = 0 \implies \operatorname{depth} A_\Delta \geq 3$.

Theorem (Benedetti, Di Marca, _, 2017)

Yes. Moreover A_Δ can be taken as R/I where $\operatorname{Proj} R$ is a rational variety (in particular R is a domain) and I is a height 1 ideal of R.

An example

Let $d = 3$, $\delta = 2$ and $\Delta = \langle\{1,2\},\{2,3\},\{1,3\}\rangle$ (so that Δ has dimension $1 = \delta - 1$). To produce A_Δ, consider:

(i) $\mathfrak{a} = (X_0, X_1, X_2) \subseteq S = K[X_0, X_1, X_2, X_3]$ ($\mathfrak{a}$ is the defining ideal of the point $P = [0,0,0,1] \in \mathbb{P}^3$);

(ii) $R = K[\mathfrak{a}_3] = K[S_3 \setminus \{X_3^3\}] \subseteq S$ ($\operatorname{Proj} R = X$ is the blow up of $\mathbb{P}^3$ along the point P);

(iii) $\mathfrak{b} = (X_0X_1X_3) \subseteq S$ ($\mathfrak{b}$ is the defining ideal of the union of planes $H = \{X_0 = 0\} \cup \{X_1 = 0\} \cup \{X_2 = 0\} \subseteq \mathbb{P}^3\}$;

(iv) $I = \mathfrak{b} \cap R = (X_0X_1X_2,\ X_0X_1X_2X_3^3,\ X_0X_1X_2X_3^6) \subseteq R$ (I is the defining ideal of the strict transform of H in X);

(v) $A_\Delta = R/I$ ($\operatorname{Proj} A_\Delta$ is the strict transform of H in X).

An example, continuation

So, A_Δ is a 3-dimensional positively graded K-algebra having 3 minimal primes $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ such that:

(i) $\text{height}(\mathfrak{p}_1 + \mathfrak{p}_2) = \text{height}(\mathfrak{p}_2 + \mathfrak{p}_3) = \text{height}(\mathfrak{p}_1 + \mathfrak{p}_3) = 1$;

(ii) $\text{height}(\mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3) = 3$.

In particular $\widetilde{H^1}(\mathcal{N}(A_\Delta); K) \neq 0$; so, since $A_\Delta = R/I$ where R is a 4-dimensional domain, if I was principal (even up to radical), this would provide a negative answer to the question of Lyubeznik. Unfortunately, this is not the case .....

Theorem

Let R be a domain of dimension at least 4 such that $X = \text{Proj}\, R$ is a rational variety, and assume that K has characteristic 0. Then, whenever $x \in R_i$ with $i > 0$, $\widetilde{H^1}(\mathcal{N}(R/xR); K) = 0$.

The question of Lyubeznik

Question (Lyubeznik)

If R is a domain of dimension at least 4, is it true that $\widetilde{H}^1(\mathcal{N}(R/xR);K) = 0$ whenever $x \in R_i$ with $i > 0$?

In view of the previous slide, in order to answer negatively the above question one has to seek for different examples. The following simple observation is crucial:

Lemma

If Lyubeznik's question had an affirmative answer, then the following would be true:
If $\mathfrak{p} \subseteq S = K[X_1,\ldots,X_n]$ *is prime,* $\dim S/\mathfrak{p} \geq 3$, *and* $\mathrm{in}(\mathfrak{p}) = I_\Delta$ *is square-free for some monomial order on* S, *then* $\widetilde{H}^1(\Delta;K) = 0$.

Proof of the lemma

In fact, just consider $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ s. t. $\mathrm{in}(\mathfrak{p}) = \mathrm{in}_w(\mathfrak{p})$. Then, if $\mathfrak{p}_{\mathrm{hom},w} \subseteq S[t]$ is the homogenization of $\mathfrak{p}$ w.r.t. w,

$$R = S[t]/\mathfrak{p}_{\mathrm{hom},w}$$

is a domain of dimension ≥ 4, and

$$R/xR \cong S/\mathrm{in}(\mathfrak{p}) = K[\Delta],$$

where $x = \overline{t} \in R$. So Δ is homotopically equivalent to $\mathcal{N}(R/xR)$; in particular

$$\widetilde{H^1}(\Delta; K) \cong \widetilde{H^1}(\mathcal{N}(R/xR; K)).$$

The latter, if Lyubeznik's question was true, would vanish. $\square$

A related question

The conclusion of the lemma is suspicious! Indeed it is not true, and I want to spend the last slides by constructing a counterexample. First let me quickly discuss a related question:

Question

If $\mathfrak{p} \subseteq S = K[X_1, \ldots, X_n]$ is prime and $\mathrm{in}(\mathfrak{p})$ is square-free, then $S/\operatorname{in}(\mathfrak{p})$ is Cohen-Macaulay (in particular $S/\mathfrak{p}$ is Cohen-Macaulay)?

The answer is no, but the question is not a complete nonsense:

Yes in the following cases:

- $\dim(S/\mathfrak{p}) \leq 2$ (Kalkbrener-Sturmfels).
- $\mathfrak{p}$ is a Cartwright-Sturmfels ideal (Brion).

Let $S = K[X_0, X_1, X_2]$, $T = K[Y_0, Y_1]$,

$$f = X_0X_1X_2 + X_1^3 + X_2^3 \in S.$$

Then S/fS is the coordinate ring of a (singular) elliptic curve E in $\mathbb{P}^2$, and $(S/fS)\sharp T$ is the coordinate ring of the Segre product

$$E \times \mathbb{P}^1 \subseteq \mathbb{P}^5.$$

So, $(S/fS)\sharp T \cong P/\mathfrak{p}$ where $P = K[Z_{i,j} : i = 0, 1,\ j = 0, 1, 2]$ and

$$\mathfrak{p} = (f(Z_{i_0,0}, Z_{i_1,1}, Z_{i_2,2}),\ [12]_Z,\ [13]_Z,\ [23]_Z : 1 \geq i_0 \geq i_1 \geq i_2 \geq 0) \subseteq P.$$

So $\mathfrak{p}$ is a prime ideal and $\dim P/\mathfrak{p} = 3$.

Notice that, if we choose the lexicographic term order on S extending $X_0 > X_1 > X_2$, we have

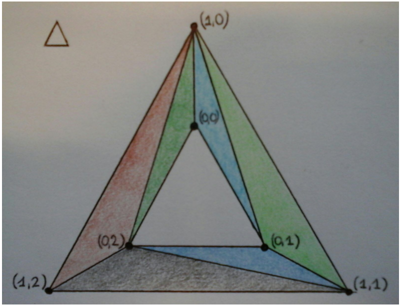
$$\mathrm{in}(f) = X_0X_1X_2.$$

By choosing the lexicographic term order on P extending $Z_{0,0} > Z_{0,1} > Z_{0,2} > Z_{1,0} > Z_{1,1} > Z_{1,2}$, one can show that the initial ideal of $\mathfrak{p}$ is the expected one:

$$\mathrm{in}(\mathfrak{p}) = (Z_{i_0,0}Z_{i_1,1}Z_{i_2,2},\ Z_{0,0}Z_{1,1},\ Z_{0,0}Z_{1,2},\ Z_{0,1}Z_{1,2} : 1 \geq i_0 \geq i_1 \geq i_2 \geq 0).$$

So $P/\,\mathrm{in}(\mathfrak{p}) = K[\Delta]$ where .....

Counterexample



In particular, $\widetilde{H^1}(\Delta; K) \neq 0$. So we get a counterexample to Lyubeznik's question, and also a nonCohen-Macaulay prime ideal with square-free initial ideal !

Degenerated Segre product

Let $S = K[X_1, \ldots, X_n]$, $T = K[Y_1, \ldots, Y_m]$, and

$$I = (u_1, \ldots, u_s) \subseteq S, \; J = (v_1, \ldots, v_t) \subseteq T$$

be monomial ideals. Then, we define the monomial ideal $I * J \subseteq P = K[Z_{i,j} : i = 1, \ldots, m, \; j = 1, \ldots, n]$ generated by:

- $u_i(Z_{i_1,1}, \ldots, Z_{i_n,n})$ s. t. $i = 1, \ldots, s$, $m \geq i_1 \geq \cdots \geq i_n \geq 1$.
- $v_j(Z_{1,j_1}, \ldots, Z_{m,j_m})$ s. t. $j = 1, \ldots, t$, $n \geq j_1 \geq \cdots \geq j_m \geq 1$.
- $Z_{i_1,j_1} Z_{i_2,j_2}$ s. t. $1 \leq i_1 < i_2 \leq m$ and $1 \leq j_1 < j_2 \leq n$.

Notice that if I and J are square-free, then $I * J$ is square-free as well; so the **degenerated Segre product** yields an operation on simplicial complexes.

Degenerated Segre product

It is not difficult to check that, if $I = \text{in}(\mathfrak{a})$ and $J = \text{in}(\mathfrak{b})$ for suitable term orders and ideals $\mathfrak{a} \subseteq S$ and $\mathfrak{b} \subseteq T$:

then $I * J = \text{in}(\mathfrak{c})$ where $(S/\mathfrak{a})\sharp(T/\mathfrak{b}) = P/\mathfrak{c}$.

Therefore in this way one can produce a lot of ideals having a square-free initial ideal, so it might be worth to inquire on the homological properties of the degenerated Segre product and compare them with the ones of the ordinary Segre product.

THANK YOU !