Betti tables of ideals in a polynomial ring

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From a joint work Jürgen Herzog and Leila Sharifan

BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

Free resolutions

Let $S = K[x_0, \ldots, x_n]$ be a polynomial ring over a field K and $I \subset S$ a graded ideal. The minimal graded free resolution of I is:

$$0 \to \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{n,j}} \to \ldots \to \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{1,j}} \to \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{0,j}} \to I \to 0.$$

The invariants $\beta_{i,j}(I) = \beta_{i,j}$ are the graded Betti numbers of I.

If $X \subset \mathbb{P}^n$ is a projective scheme, we will refer to its free resolution (and related concepts) as the one of the saturated ideal defining it.

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The Betti table of I is the matrix $(\beta_{i,i+d}(I))_{i,d}$. It can be thought as a $(n+1) \times \operatorname{reg}(I)$ matrix: For example, if

$$I = (x_0x_1, x_1x_2, x_2x_3, x_3^2) \subset S = K[x_0, \dots, x_3],$$

the resolution of I is:

$$0 \to S(-4) \to S(-3)^3 \oplus S(-4) \to S(-2)^4 \to I \to 0$$

Therefore its Betti table is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

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Ideals with linear resolution

The ideal I is said to have d-linear resolution if all its minimal generators are of degree d and $\mathrm{reg}(I) = d$. Equivalently, if the Betti tables of I has only one nonzero row, the dth.

For example, the rational normal curve in $\mathbb{P}^4$ has Betti table:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 6 & 8 & 3 & 0 & 0 \end{pmatrix}$$

thus it has 2-linear resolution. Indeed it is known that all varieties of minimal degree have a 2-linear resolution. More generally:

(Bruns, Conca, -). If I defines a variety of minimal degree, then I^k has $2k$-linear resolution for each k.

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A graded Betti number $\beta_{i,i+d}$ of I is said extremal if $\beta_{i,i+d}(I) \neq 0$ and $\beta_{h,h+k}(I) = 0$ for all $(h,k) \neq (i,d)$ with $h \geq i$ and $k \geq d$.

For example, let us look at the Betti table of

$$I = (x_0^4, x_0^3x_1, x_0^3x_2, x_0^3x_3, x_0^2x_1^3, x_0x_1^4, x_0^2x_1^2x_2, x_0x_1^3x_2, x_1^6)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 6 & 4 & 1 \\ 4 & 6 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

The extremal Betti numbers of I are those marked in red.

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Cohomological interpretation

Let $X \subset \mathbb{P}^n$ be a projective scheme and $\mathcal{I}_X$ its ideal sheaf. Then $\beta_{i,i+d}$ is an extremal Betti number of $\bigoplus_{m \in \mathbb{Z}} \Gamma(X, \mathcal{I}_X(m)) \subset S$ iff:

(i) $i < n$;

(ii) $\dim_K(H^p(X, \mathcal{I}_X(q-p))) = \beta_{i,i+d} \neq 0$ for $p = n - i$ and $q = d - 1$;

(iii) $H^r(X, \mathcal{I}_X(s-r)) = 0$ for all $(r,s) \neq (p,q)$, $1 \le r \le p$ and $s \ge q$.

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C o h o m o l o g i c a l   i n t e r p r e t a t i o n

Let $X \subset \mathbb{P}^n$ be a projective scheme and $\mathcal{I}_X$ its ideal sheaf. Then $\beta_{i,i+d}$ is an extremal Betti number of $\bigoplus_{m\in\mathbb{Z}} \Gamma(X, \mathcal{I}_X(m)) \subset S$ iff:

(i) $i < n$;

(ii) $\dim_K(H^p(X, \mathcal{I}_X(q-p))) = \beta_{i,i+d} \neq 0$ for $p = n - i$ and $q = d - 1$;

(iii) $H^r(X, \mathcal{I}_X(s-r)) = 0$ for all $(r, s) \neq (p, q)$, $1 \leq r \leq p$ and $s \geq q$.

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Goals of the talk

We will explain how to give a numerical characterization of:

(i) The Betti tables of ideals $I \subset S$ with d-linear resolution.

(ii) The extremal Betti numbers of any graded ideal $I \subset S$.

For simplicity, we will assume that the characteristic of K is 0.

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Reduction to Borel-fixed ideals

If I has d-linear resolution, then it has the same Betti table of its generic initial ideal $\mathrm{Gin}(I)$ (Aramova, Herzog, Hibi).

$\mathrm{Gin}(I)$ is strongly stable ($\mathrm{Gin}(I) : x_i \subset \mathrm{Gin}(I) : x_j \ \ \forall\, j < i$).

Viceversa, any strongly stable ideal generated in degree d has d-linear resolution (Elihaou, Kervaire).

So we are allowed to focus on the Betti tables of strongly stable monomials ideals $J \subset S$ generated in degree d.

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The Elihaou-Kervaire formula

Given an ideal $I \subset S = K[x_0, \ldots, x_n]$, we define:

$$\sum_{k=0}^{n} m_k(I) t^k = \sum_{i=0}^{n} \beta_i(I)(t-1)^i.$$

Obviously to characterize the possible Betti tables of ideals with linear resolution we can characterize the possible sequences $(m_0(I), m_1(I), \ldots, m_n(I))$.

For a monomial $u \in S$, let us set $m(u) = \max\{i : x_i | u\}$. Elihaou and Kervaire showed that, if $J \subset S$ is strongly stable, then:

$$m_k(J) = |\{u \in G(J) : m(u) = k\}|$$

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A multiplicative structure on S_d

Given two monomials $u, v \subset S_d$, write them as $u = x_{i_1} \cdots x_{i_d}$ and $x_{j_1} \cdots x_{j_d}$ with $i_1 \leq \ldots \leq i_d$ and $j_1 \leq \ldots \leq j_d$, and define:

$$u * v = \begin{cases} x_{i_1+j_1} x_{i_2+j_2} \cdots x_{i_d+j_d} & \text{if } i_d + j_d \leq n, \\ 0 & \text{otherwise} \end{cases}$$

We extend the operation to the whole S_d by K-linearity, and denote by S_d the gotten K-algebra. For example, if $d = 4$, $n \geq 6$, $u = x_0 x_1^2 x_3$ and $v = x_2^2 x_3^2$, then:

$$u * v = x_0 x_1 x_1 x_3 * x_2 x_2 x_3 x_3 = x_2 x_3 x_4 x_6.$$

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The ring $\mathcal{S}_d$

Notice that $\mathcal{S}_d$ has a natural $\mathbb{N}$-grading, namely $\mathcal{S}_d = \bigoplus_{k=0}^{n} (\mathcal{S}_d)_k$,

$$(\mathcal{S}_d)_k = \langle u \in S_d : m(u) = k \rangle.$$

One can show that there is a graded isomorphism of K-algebras:

$$\mathcal{S}_d \cong \frac{K[y_1, \ldots, y_d]}{(y_1, \ldots, y_d)^{n+1}}.$$

We showed that, if $J \subset S$ is a strongly stable monomial ideal, then $\mathcal{G}(J) = (G(J), *)$ is a quotient of $\mathcal{S}_d$. The Hilbert function of $\mathcal{G}(J)$ is:

$$\dim_K(\mathcal{G}(J)_k) = m_k(J),$$

so $(m_0(J), m_1(J), \ldots, m_n(J))$ satisfies Macaulay's conditions.

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One can show that there is a graded isomorphism of K-algebras:

$$\mathcal{S}_d \cong \frac{K[y_1, \ldots, y_d]}{(y_1, \ldots, y_d)^{n+1}}.$$

We showed that, if $J \subset S$ is a strongly stable monomial ideal, then $\mathcal{G}(J) = (\mathrm{G}(J), *)$ is a quotient of $\mathcal{S}_d$. The Hilbert function of $\mathcal{G}(J)$ is:

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BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

The ring $\mathcal{S}_d$

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(i) There exists an ideal $I \subset S$ with d-linear resolution such that $m_k(I) = m_k$ for all $k = 0, \ldots, n$.

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BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

Betti numbers of ideals with linear resolution

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BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

Example

We will never find an ideal $I \subset S$ with the below minimal free resolution:

$$0 \to S(-6)^6 \to S(-5)^{22} \to S(-4)^{29} \to S(-3)^{14} \to I \to 0.$$

Indeed we would have $m_0(I) = 1$, $m_1(I) = 3$, $m_2(I) = 4$ and $m_3(I) = 6$,

but $m_2(I)^{\langle 2 \rangle} = 4^{\langle 2 \rangle} = 5 < 6 = m_3(I)$.

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Our initial dream was to characterize the possible Betti tables of componentwise linear ideals $I \subset S$, that is such that $I_{\langle m \rangle}$ has m-linear resolution for all m, where $I_{\langle m \rangle} = (f \in I : \deg(f) = m)$. The interest in this comes from the fact that the generic initial ideal of every homogeneous ideal is componentwise linear.

Our characterization of the Betti tables of ideals with linear resolution gives some necessary conditions that a Betti table of a componentwise linear ideal must satisfy. We show that these conditions are also sufficient up to three variables. Unfortunately, this is no longer true when the number of variables is bigger.

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BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

Componentwise linear ideals

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BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

Betti tables do not detect componentwise linearity

Let us consider the following two ideals in $K[x_0, x_1, x_2]$:

$$I = (x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_1^4, x_0^3x_2, x_0^2x_1x_2^2, x_0^2x_2^3, x_0x_1^2x_2^2),$$
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One can check that I is componentwise linear, whereas J is not.

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BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

Characterization of extremal Betti numbers

If we restrict our attention to the extremal Betti numbers of a componentwise linear ideal, we are able to show that "the necessary conditions discussed above become sufficient"!

Exploiting a result of Bayer, Charalambous and Popescu, this leads to a numerical characterization of the positive integers:

(i) $0 < i_1 < i_2 < \ldots < i_k \leq n$

(ii) $d_1 > d_2 > \ldots > d_k > 0$

(iii) $b_1, b_2, \ldots, b_k$

such that exists a graded ideal $I \subset S$ with extremal Betti numbers:

$$\beta_{i_p, i_p + d_p}(I) = b_p \quad \forall\ p = 1, \ldots, k$$

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BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

Characterization of extremal Betti numbers

If we restrict our attention to the extremal Betti numbers of a componentwise linear ideal, we are able to show that "the necessary conditions discussed above become sufficient"!

Exploiting a result of Bayer, Charalambous and Popescu, this leads to a numerical characterization of the positive integers:

(i) $0 < i_1 < i_2 < \ldots < i_k \leq n$

(ii) $d_1 > d_2 > \ldots > d_k > 0$

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BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

Picture

$$\begin{pmatrix}
* & * & * & \dots & * & * & \dots & * & * & \dots \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots\vdots\vdots \\
* & * & * & \dots & * & * & \dots & \beta_{i_3,i_3+d_3} & 0 & \dots \\
* & * & * & \dots & * & 0 & \dots & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots\vdots\vdots \\
* & * & * & \dots & \beta_{i_2,i_2+d_2} & 0 & \dots & 0 & 0 & \dots \\
* & * & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots\vdots\vdots \\
* & \beta_{i_1,i_1+d_1} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots
\end{pmatrix}$$

BETTI TABLES OF IDEALS IN A POLYNOMIAL RING

Picture

$$\begin{pmatrix}
\ast & \ast & \ast & \dots & \ast & \ast & \dots & \ast & \ast & \dots \\
\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots \\
\ast & \ast & \ast & \dots & \ast & \ast & \dots & \beta_{i_3,i_3+d_3} & 0 & \dots \\
\ast & \ast & \ast & \dots & \ast & 0 & \dots & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots \\
\ast & \ast & \ast & \dots & \beta_{i_2,i_2+d_2} & 0 & \dots & 0 & 0 & \dots \\
\ast & \ast & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots \\
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\end{pmatrix}$$

The above numerical characterization requires some technical definitions, so I prefer to quickly explain how we could get it: In the proof of our result on the Betti numbers of ideals with linear resolution, we actually construct a special strongly stable ideal generated in degree d, termed piecewise lexsegment, with prescribed Betti numbers. One of the obstructions to characterize the Betti tables of componentwise linear ideals, is that a piecewise lexsegment ideal does not keep its special feature when multiplied with the maximal ideal. Many piecewise lexsegment ideals have the same extremal Betti number (in this case there is only the one given by the projective dimension). Roughly speaking, we are able to choose a special one among them, which keeps enough of its properties when multiplied with the maximal ideal.

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Sketch of the proof

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