COHOMOLOGICAL DIMENSION, CONNECTEDNESS PROPERTIES AND INITIAL IDEALS

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such that:

$$\sqrt{LT_{\prec}(I)} = J \ ?$$

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In fact the h-vector of P/J is admissible, since

$$HS_{P/J}(z) = \frac{h(z)}{(1-z)^2} = \frac{1+4z+2z^2}{(1-z)^2}$$

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$$I \text{ prime} \Rightarrow P/LT_\prec(I) \text{ connected in codimension } 1$$

References

References

- [HuTa, 2004] C. Huneke, A. Taylor, *Lectures on Local Cohomology*, notes for students of the University of Chicago, 2004.

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- [V, 2008] M. Varbaro, *Cohomological dimension, connectedness properties and initial ideals*, arXiv:0802.1800, 2008.

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Translate this result from *complete* to *graded* case.

- Applications to initial ideals

We will use the first part to prove the main result.

PART 1

LOCAL COHOMOLOGY AND CONNECTEDNESS

Notations, definitions and "basic" results

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- If R is local or a polinomial ring , then

$$\mathrm{ht}(\mathfrak{a}) \leq \mathrm{cd}(R;\mathfrak{a}) \leq \mathrm{ara}(\mathfrak{a}) \leq \dim R.$$

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T is connected if and only if $\mathrm{c}(T) \geq 0$.

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If T is irreducible, then $\mathrm{c}(T) = \dim T$.

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EXAMPLES

$$\mathfrak{a} = (xz, xw, yz, yw) \subseteq \mathbb{C}[x, y, z, w],\ T = \mathcal{Z}(\mathfrak{a}) \subseteq \mathbb{A}^4.$$

$$T = \mathcal{Z}(x, y) \cup \mathcal{Z}(z, w), \text{ and } \mathcal{Z}(x, y) \cap \mathcal{Z}(z, w) = \{(0, 0, 0, 0)\},$$

so $\dim T = 2$ and $\mathrm{c}(T) = 0$.

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If $T = \operatorname{Spec} R$, then $\mathrm{c}(R) := \mathrm{c}(T)$ is the infimum of $\dim R/\mathfrak{a}$ with $\mathfrak{a}$ ideal such that $\operatorname{Spec} R \setminus \mathcal{V}(\mathfrak{a})$ is disconnected.

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$S = k[x, y, z, v, w], \mathfrak{a} = (xy, xv, xw, yv, yz, vz, wz), R = S/\mathfrak{a}.$

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Corollary *(Hochster and Huneke)*

$(R, \mathfrak{m})$ *complete, loc., equidimensional,* $H^{\dim R}_{\mathfrak{m}}(R)$ *indecomposable.*
If $\mathrm{cd}(R, \mathfrak{a}) \leq \dim R - 2$ *then* $\mathrm{Spec}\, R/\mathfrak{a} \setminus \{\mathfrak{m}\}$ *is connected.*

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- If $R \longrightarrow S$, then $\mathrm{ara}_R(\mathfrak{a}) \geq \mathrm{ara}_S(\mathfrak{a}S)$

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from which $\mathrm{cd}(R, \mathfrak{a} + \mathfrak{b}) \leq \mathrm{cd}(R, \mathfrak{a}) + \mathrm{cd}(R, \mathfrak{b})$!

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PART 2

APPLICATIONS TO INITIAL IDEALS

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- $LT_{\prec}(I) \subseteq P$ ideal of leading terms of I with respect to $\prec$;
- $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{Z}_+)^n$ weight vector;

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Corollary *(Kalkbrener and Sturmfels, [KaSt, 1995]).*

For every ω and I, if I is prime, then

$P/\operatorname{in}_\omega(I)$ is connected in codimension 1.

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$P/\operatorname{in}_\omega(I)$ *is connected in codimension 1.*

Proof of Corollary

The main result

For every I and ω

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Proof of Corollary

One can prove that $\mathrm{c}(P/I) \geq \mathrm{depth}(P/I) - 1$.

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Graded version of the main result of the first part let us conclude!

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The minimal primes of $J \subseteq P := k[x, y, z, u, v, w, a]$ are:

$$\wp_1 = (x, y, z, u)\ ,\ \wp_2 = (x, y, v, a),\ \wp_3 = (x, z, v, a),$$

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Note that $\dim P/(\wp_1 + \wp_i) = 1$ whereas $\dim P/J = 3$.

So P/J is not connected in codimension 1, therefore

cannot exist $I \subseteq P$ Cohen-Macaulay and $\prec$ s. t. $\sqrt{LT_{\prec}(I)} = J$!