RELATIONS BETWEEN MINORS

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The QUADRATIC equation above is called Plücker relation.

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What about the case $t < m$?

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In this talk we are going to discuss the above problem.

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-) For any $1 \leq i_1 < \ldots < i_t \leq m$ and $1 \leq j_1 < \ldots < j_t \leq n$

$$[i_1, \ldots, i_t | j_1, \ldots, j_t] = \det \begin{pmatrix} x_{i_1 j_1} & \cdots & x_{i_1 j_t} \\ \vdots & \ddots & \vdots \\ x_{i_t j_1} & \cdots & x_{i_t j_t} \end{pmatrix}.$$

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$$[i_1, \ldots, i_t | j_1, \ldots, j_t] \mapsto \overline{(f_{i_1} \wedge \ldots \wedge f_{i_t}) \otimes (e_{j_1} \wedge \ldots \wedge e_{j_t})}$$

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-) $d(2, 3, n) = d(2, 4, n) = 3$ (whenever $n \geq 4$).

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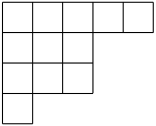

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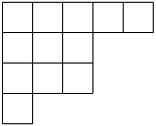

A standard tableu of shape λ is a filling of the boxes of λ which is rows increasing and columns nondecreasing.

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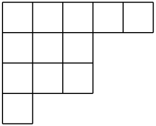
$\lambda = (5, 3, 3, 1) =$ 

A standard tableu of shape λ is a filling of the boxes of λ which is rows increasing and columns nondecreasing. For instance

$T =$

1	3	6	9	10
2	5	7		
2	6	7		
3				

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$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 \\ \cline{3-3}\end{array} \quad \Bigg| \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \rightsquigarrow \qquad [1,3,4|2,3,5] \cdot [3|2]$$

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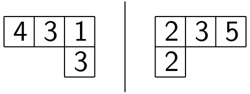

$$\rightsquigarrow \qquad [1, 3, 4 | 2, 3, 5] \cdot [3 | 2]$$

It turns out that the $\Bbbk$-subspace of S generated by the products of minors of shape λ is isomorphic as G-module to $L_\lambda W \otimes L_\lambda V$.

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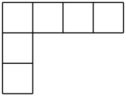

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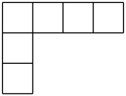

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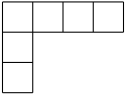

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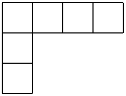

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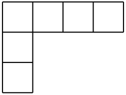

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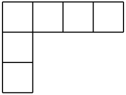

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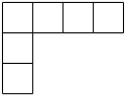
$\lambda = (4, 1, 1) =$ 

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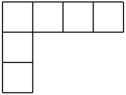

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So we will face the problem looking at $K_t = K_t(m,n) = \mathrm{Ker}(\psi)$.

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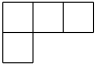
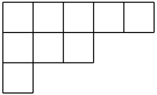
$\lambda =$  and $\tilde{\lambda} =$ 

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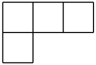
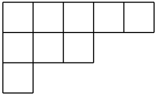
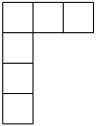
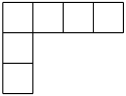
$\lambda =$  and $\tilde{\lambda} =$ 

Thus the following γ is a successor of λ, whether μ is not

$\mu =$  and $\gamma =$ 

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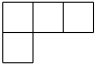
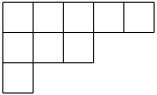
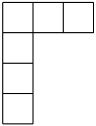
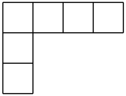
$\lambda =$  and $\tilde{\lambda} =$ 

Thus the following γ is a successor of λ, whether μ is not

$\mu =$  and $\gamma =$ 

Pieri's formula yields the isomorphism of $\mathrm{GL}(W)$-modules

$$L_\lambda W \otimes \bigwedge^t W \cong \bigoplus_{\mu \text{ successor of } \lambda} L_\mu W.$$

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$$b(\lambda,\mu) = \sum_{\substack{\lambda' \text{ predecessor of } \lambda \\ \mu' \text{ predecessor of } \mu}} b(\lambda',\mu').$$

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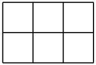
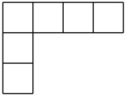
$\lambda_0 =$  and $\mu_0 =$ 

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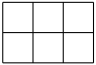
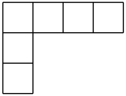
$\lambda_0 =$  and $\mu_0 =$ 

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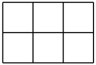
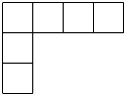
$\lambda_0 =$  and $\mu_0 =$ 

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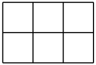
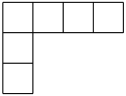
$\lambda_0 =$  and $\mu_0 =$ 

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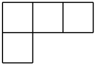
$\gamma_0 = (3,1) =$ 

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corresponds to

$$\det \begin{pmatrix} [12 \mid 12] & [12 \mid 13] & [12 \mid 14] \\ [13 \mid 12] & [13 \mid 13] & [13 \mid 14] \\ [23 \mid 12] & [23 \mid 13] & [23 \mid 14] \end{pmatrix}$$

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Are quadrics and cubics enough to generate J_t and K_t?

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Since $\lambda_1 \leq m$, then $L_\lambda W \otimes L_\mu V$ is not minimal in K_t if $\mu_1 > m + t$.

On the other hand, if $\mu_1 \leq m + t$, there is a polynomial in $L_\lambda W \otimes L_\mu V$ that actually belongs in $P_t(m, m+t)$.

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So, in general, $d(t, m, n) \leq d(t, m, m + t)$.

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2-minors of a $3 \times n$ and of a $4 \times n$ matrix

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So, in a $3 \times n$ and in a $4 \times n$ matrix, the only minimal relations between 2-minors are quadrics and cubics!