

Test, multiplier and invariant ideals

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Measuring singularities

Given a polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ vanishing at $z \in \mathbb{C}^N$, by definition f is **singular** at z if $\frac{\partial f}{\partial x_i}(z) = 0 \forall i = 1, \dots, N$.

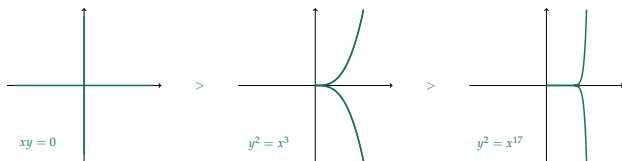
The first way to quantify how singular is f at z is by means of the **multiplicity**: The multiplicity of f at z is the largest d such that $\partial f(z) = 0$ for all differential operators ∂ of order less than d . So

f has multiplicity > 1 at $z \Leftrightarrow f$ is singular at z .

If $z = 0 \in \mathbb{C}^N$, then it is easy to see that the multiplicity of f in z is simply the degree of the lowest degree term of f .

The multiplicity is a quite rough measurement of singularities though...

Curves with multiplicity 2 at the origin



The three curves above have multiplicity 2 at the origin. However, the above singularities are evidently quite different. For today, we will consider the first singularity better than the second, which in turns will be better than the third...

Analytic approach

Given a polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$, let us consider the (almost everywhere defined) function

$$\begin{aligned} \mathbb{C}^N = \mathbb{R}^{2N} &\rightarrow \mathbb{R} \\ z &\mapsto 1/|f(z)| \end{aligned}$$

We want to measure how fast the above function blows up at a point z such that $f(z) = 0$. The faster, the worse is the singularity.

Wlog, from now on we will consider $z = 0$ (so that $f(0) = 0$). As we learnt in the first calculus class, the function is not square integrable in a neighborhood of 0, that is: the integral

$$\int_B \frac{1}{|f|^2}$$

does not converge for any neighborhood B of 0.

Analytic approach

On the other hand, if f is nonsingular at 0, then there exists a neighborhood B of 0 such that the integral

$$\int_B \frac{1}{|f|^{2\lambda}}$$

converges for all real numbers $\lambda < 1$. Does this property characterize smoothness? **NO!**

EXAMPLE: If $f = x_1^{a_1} \cdots x_N^{a_N}$, it is easy to see that

$$\int_B \frac{1}{|f|^{2\lambda}}$$

converges for a small neighborhood B of 0 iff $\lambda < \min_i \{1/a_i\}$. In particular, if f is a square-free monomial, then the above integral converges for any $\lambda < 1$, as in the smooth case!

Analytic definition

Def.: The **log-canonical threshold** of $f \in \mathfrak{m} = (x_1, \dots, x_N)$ is

$$\text{lct}(f) = \sup\{\lambda \in \mathbb{R}_{>0} : \exists \text{ a neighborhood } B \text{ of } 0 \text{ s.t. } \int_B \frac{1}{|f|^{2\lambda}} < \infty\}.$$

More generally, for each $\lambda \in \mathbb{R}_+$, the **multiplier ideal** (with coefficient λ) $\mathcal{J}(\lambda \bullet f)$ of f is the following ideal of $\mathbb{C}[x_1, \dots, x_N]$:

$$\left\{ g \in \mathbb{C}[x_1, \dots, x_N] : \exists \text{ a neighborhood } B \text{ of } 0 \text{ s.t. } \int_B \frac{|g|^2}{|f|^{2\lambda}} < \infty \right\}$$

$$\left[\begin{array}{ccccccc} \mathcal{J} = (1) & & \mathcal{J} \neq (1) & & \dots & & \dots \\ & \underbrace{\quad} & \underbrace{\quad} & & & & \\ & c_1 & c_2 & & & & \end{array} \right] \longrightarrow \lambda\text{-axis}$$

$$(1) \supsetneq \mathcal{J}(c_1 \bullet f) \supsetneq \mathcal{J}(c_2 \bullet f) \supsetneq \dots \supsetneq \mathcal{J}(c_n \bullet f) \supsetneq \dots$$

The c_i above are called **jumping numbers**. Note that $c_1 = \text{lct}(f)$.

Multiplier ideals in general

Even more generally, one can define the multiplier ideals $\mathcal{J}(\lambda \bullet I)$ (and so the jumping numbers and the log-canonical threshold) for any ideal $I = (f_1, \dots, f_r) \subseteq \mathfrak{m}$, and not just for a polynomial:

$$\left\{ g \in \mathbb{C}[x_1, \dots, x_N] : \exists \text{ a neighborhood } B \text{ of } 0 \text{ s.t. } \int_B \frac{|g|^2}{(\sum_{i=1}^r |f_i|^2)^\lambda} < \infty \right\}$$

In the words of [Lazarsfeld](#), “the intuition is that these ideals will measure the singularities of functions $f \in I$, with ‘nastier’ singularities being reflected in ‘deeper’ multiplier ideals”.

Multipliers of some invariant ideals

There are not many examples of ideals for which the multiplier ideals are known. A first class of examples was provided by [Howald](#) in 2001: He gave an explicit formula for the multiplier ideals of any [monomial ideal](#). Two years later, [Johnson](#) computed the multiplier ideals of any [determinantal ideal](#).

Recently, in a joint work with [Ines Henriques](#), we developed a general method to compute much larger classes of examples: given a vector space V of dimension N over a field K , we have a ring isomorphism:

$$S := K[x_1, \dots, x_N] \cong \text{Sym } V = \bigoplus_{d \in \mathbb{N}} \text{Sym}^d V.$$

If G is a subgroup of $\text{GL}(V)$, then, G acts naturally on S . In such a situation, the question is the following:

[What are the multiplier ideals of the homogeneous \$G\$ -invariant ones?](#)

Multipliers of some invariant ideals

- ▶ $G = \mathrm{GL}(V)$ is easy (the only homogeneous G -invariant ideals are the powers of the irrelevant ideal \mathfrak{m}):

$$\mathcal{J}(\lambda \bullet \mathfrak{m}^d) = \mathfrak{m}^{[\lambda d] + 1 - N}.$$

- ▶ Intermediate G ?????
- ▶ $G = \{1\}$ is hopeless (all the ideals are G -invariant).

Multipliers of some invariant ideals

Theorem (Henriques, -): We give an explicit description of the multiplier ideals of all the homogeneous G -invariant ideals in these cases (E and F are finite K -vector spaces and $\text{char}(K) = 0$):

- ▶ $V = E \otimes F$ and $G = \text{GL}(E) \times \text{GL}(F)$ (the G -invariant ideals are ideals of minors of a generic matrix, their products, their symbolic powers, sums of these, and more...)
- ▶ $V = \text{Sym}^2 E$ and $G = \text{GL}(E)$ (the G -invariant ideals are ideals of minors of a generic symmetric matrix, their products, their symbolic powers, sums of these, and more...)
- ▶ $V = \bigwedge^2 E$ and $G = \text{GL}(E)$ (the G -invariant ideals are ideals of pfaffians of a generic skew-symmetric matrix, their products, their symbolic powers, sums of these, and more...)
- ▶ G the group of diagonal matrices of $\text{GL}(V)$ (the G -invariant ideals here are exactly the monomial ideals).

Reduction to positive characteristic

How did we do? Actually, we proved more, in fact we developed a strategy to compute all the (generalized) test ideals of suitable ideals in characteristic $p > 0$ and by a result of Hara-Yoshida their “limit as $p \rightarrow \infty$ ” will be the multiplier ideal.

Let me notice that the reduction to characteristic p in this situation is quite surprising, since the G -invariant ideals in positive characteristic are not well-understood, e.g. in the first case (whereas in characteristic 0 they are by the work of De Concini, Eisenbud and Procesi).

Let me say that the F -pure threshold of determinantal ideals had already been computed by Miller, Singh, - in 2012, and the test ideals of ideals of maximal minors of a generic matrix by Henriques in 2014.

F -pure threshold

Let K be a field of characteristic $p > 0$, $S = K[x_1, \dots, x_N]$, $f \in S$ vanishing at 0 and e be a positive integer. Define

$$\nu_f(e) = \max\{s \in \mathbb{N} : f^s \notin (x_1^{p^e}, \dots, x_N^{p^e})\}$$

We have $0 \leq \nu_f(e) \leq p^e$ and $\nu_f(e+1) \geq p \cdot \nu_f(e)$, so the sequence $\{\nu_f(e)/p^e\}_{e \in \mathbb{N}} \subseteq [0, 1]$, being monotone, admits a limit.

The F -pure threshold of f is:

$$\text{fpt}(f) = \lim_{e \rightarrow \infty} \nu_f(e)/p^e.$$

If $g \in \mathbb{Z}[x_1, \dots, x_N]$ we denote by g_p and by g_0 the images of g in $\mathbb{Z}/p\mathbb{Z}[x_1, \dots, x_N]$ and, respectively, in $\mathbb{C}[x_1, \dots, x_N]$.

Hara-Yoshida: $\lim_{p \rightarrow \infty} \text{fpt}(g_p) = \text{lct}(g_0)$.

Thanks!

Unfortunately my time is over, but if you want to know more ask me, Ines, or read our paper: I.B. Henriques, M. Varbaro, *Test, multiplier and invariant ideals*, arXiv:1407.4324.