RELATIONS AMONG MINORS

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What about the case $t < m$?

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In this talk we are going to discuss the above problem.

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-) For any $1 \leq i_1 < \ldots < i_t \leq m$ and $1 \leq j_1 < \ldots < j_t \leq n$

$$[i_1, \ldots, i_t | j_1, \ldots, j_t] = \det \begin{pmatrix} x_{i_1 j_1} & \cdots & x_{i_1 j_t} \\ \vdots & \ddots & \vdots \\ x_{i_t j_1} & \cdots & x_{i_t j_t} \end{pmatrix}.$$

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$$[i_1, \ldots, i_t | j_1, \ldots, j_t] \mapsto \overline{(f_{i_1} \wedge \ldots \wedge f_{i_t}) \otimes (e_{j_1} \wedge \ldots \wedge e_{j_t})}$$

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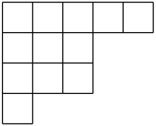

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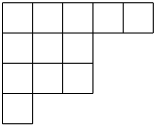

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$$T \;=\; \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 6 & 9 & 10 \\ \hline 2 & 5 & 7 \\ \cline{1-3} 2 & 6 & 7 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

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$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline \multicolumn{2}{c|}{} & 3 \\ \cline{3-3}\end{array} \;\Bigg|\; \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \rightsquigarrow \qquad [1,3,4|2,3,5] \cdot [3|2]$$

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To a pair of standard tableux of shape λ we can associate a product of minors as in the example below

$$\begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline \multicolumn{2}{c|}{} & 3 \\ \cline{3-3} \end{array} \quad \Bigg| \quad \begin{array}{|c|c|c|} \hline 2 & 3 & 5 \\ \hline 2 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \qquad \rightsquigarrow \qquad [1,3,4|2,3,5] \cdot [3|2]$$

It turns out that the k-subspace of S generated by the products of minors of shape λ is isomorphic as G-module to $L_\lambda W \otimes L_\lambda V$.

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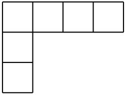

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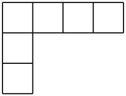

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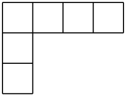

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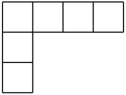
$\lambda \ = \ (4, 1, 1) \ =$ 

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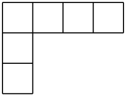

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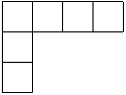

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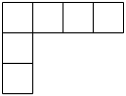

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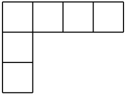

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So we will face the problem looking at $K_t = K_t(m, n) = \mathrm{Ker}(\psi)$.

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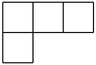
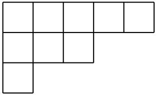
$\lambda =$  and $\tilde{\lambda} =$ 

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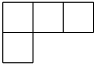
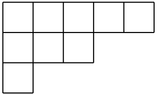
$\lambda =$  and $\tilde{\lambda} =$ 

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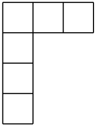
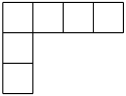
$\mu =$  and $\gamma =$ 

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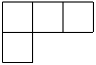
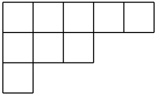
$\lambda =$  and $\tilde{\lambda} =$ 

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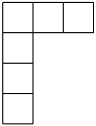
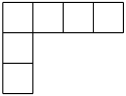
$\mu =$  and $\gamma =$ 

Pieri's formula yields the isomorphism of $\mathrm{GL}(W)$-modules

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-) If $\lambda,\mu \vdash dt$ with $d > 1$, then

$$b(\lambda,\mu) = \sum_{\substack{\lambda' \text{ predecessor of } \lambda \\ \mu' \text{ predecessor of } \mu}} b(\lambda',\mu').$$

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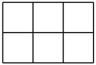
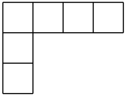
$\lambda_0 =$  and $\mu_0 =$ 

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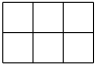
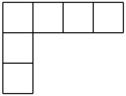
$\lambda_0 =$  and $\mu_0 =$ 

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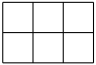
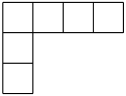
$\lambda_0 =$  and $\mu_0 =$ 

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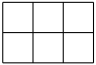
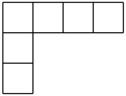
$\lambda_0 =$  and $\mu_0 =$ 

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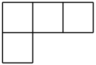
$\gamma_0 = (3, 1) =$ 

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$$L_{\lambda_0} W \otimes L_{\mu_0} V \text{ is minimal in } K_2$$

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corresponds to

$$\det \begin{pmatrix} [12 \mid 12] & [13 \mid 13] & [12 \mid 23] \\ [12 \mid 13] & [13 \mid 13] & [13 \mid 23] \\ [12 \mid 14] & [13 \mid 14] & [23 \mid 14] \end{pmatrix}$$

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Are quadrics and cubics enough to generate J_t and K_t?

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Since $\lambda_1 \leq m$, then $L_\lambda W \otimes L_\mu V$ is minimal in K_t whenever $\mu_1 > m + t$.

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Since $\lambda_1 \leq m$, then $L_\lambda W \otimes L_\mu V$ is minimal in K_t whenever $\mu_1 > m + t$.

On the other hand, if $\mu_1 \leq m + t$, there is a polynomial in $L_\lambda W \otimes L_\mu V$ that actually lies in $P_t(m, m + t)$.

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For instance if $t = 2$ and $m = 4$, $L_\lambda W \otimes L_\mu V$, where $\lambda = (4, 3, 1)$ and $\mu = (6, 2)$, might be minimal in K_2.

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$$\begin{array}{cccc} 4 & 3 & 2 & 1 \\ & 3 & 2 & 1 \\ & & & 1 \end{array} \quad \Bigg| \quad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & & & & \\ & & & & & \end{array}$$

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In fact F is in the variables $(f_{i_1} \wedge f_{i_2}) \otimes (e_{j_1} \wedge e_{j_2})$ with $i_2 \leq 4$ and $j_2 \leq 6$.

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\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\
\cline{4-4}
\end{array}
\quad\Bigg|\quad
\begin{array}{|c|c|c|c|c|c|}
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\hline
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So, in general, $d(t, m, n) \leq d(t, m, m + t)$.

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So, in a $(3 \times n)$-matrix, “essentially” the only relations among 2-minors are quadrics and cubics!