Cohomological Dimension, Symbolic Powers and Matroids

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PAPERS AND STRUCTURE OF THE THESIS

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ZARISKI TOPOLOGY

VS

ÉTALE TOPOLOGY

Let $S := \Bbbk[x_0, \ldots, x_n]$ be the polynomial ring in $n+1$ variables over an algebraically closed field $\Bbbk$, $f_1, \ldots, f_r$ homogeneous polynomials of S and

$$X := \{P \in \mathbb{P}^n : f_1(P) = \ldots = f_r(P) = 0\} \subseteq \mathbb{P}^n.$$

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ZARISKI TOPOLOGY VS ÉTALE TOPOLOGY

Some results to deal with this problem

(Eisenbud, Evans, 1972): $X \neq \emptyset \implies \operatorname{ara}(X) \leq n$.

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For any scheme U, we define the cohomological dimension of U as:

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$$\text{écd}(\mathbb{P}^n \setminus X) > \mathrm{cd}(\mathbb{P}^n \setminus X) + n$$

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Very sketchy version of the proof

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SYMBOLIC POWERS

AND

MATROIDS

Let $S := \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\Bbbk$.

(Cowsik, Nori, 1976): Given $I \subseteq S$ homogeneous and radical, then S/I^k is Cohen-Macaulay $\forall\ k \iff I$ is a complete intersection.

S/I^k CM $\implies I^k$ is equal to the kth symbolic power $I^{(k)}$ of I.

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QUESTION: When is $S/I^{(k)}$ Cohen-Macaulay for all k??

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We can supply an answer to the above question when I is a square-free monomial ideal.

Set $[n] := \{1, \ldots, n\}$ and recall the bijection of sets

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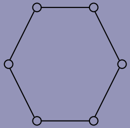

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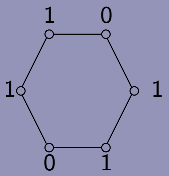

1-cover

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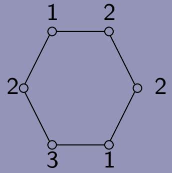
EXAMPLES: 

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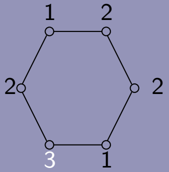

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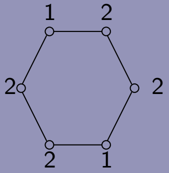
EXAMPLES: 

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$$\bar{A}(\Delta) := \bigoplus_{k \in \mathbb{N}} J(\Delta)^{(k)} / \mathfrak{m} J(\Delta)^{(k)}, \ \mathfrak{m} = (x_1, \ldots, x_n) \subseteq S.$$

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THANKS FOR YOUR ATTENTION