RELATIONS BETWEEN THE MINORS OF A GENERIC MATRIX

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ABSTRACT. It is well-known that the Plücker relations generate the ideal of relations of the maximal minors of a generic $m \times n$ matrix. In this paper we discuss the relations of t-minors for $t < \min(m,n)$. We will exhibit minimal relations in degrees 2 (non-Plücker in general) and 3, and give some evidence for our conjecture that we have found the generating system of the ideal of relations. The approach is through the representation theory of GL.

INTRODUCTION

In algebra, in algebraic geometry and in representation theory the polynomial relations between the minors of a matrix are interesting objects for many reasons. Surprisingly they are still unknown in almost all cases. While it is a classical theorem that the Plücker relations (of maximal minors of a generic matrix) generate the defining ideal of the Grassmannian, only a few other cases have been treated, for example, the principal minors of a (symmetric) matrix, see Holtz and Sturmfels [13], Lin and Sturmfels [15] and Oeding [16]. For arbitrary t, the relations between the t-minors of a generic matrix are certainly not understood, and in this paper we try to investigate them.

We refer the reader to Fulton and Harris [12], Procesi [17], and Weyman [18] for background in representation theory, to Bruns and Vetter [7] for the theory of determinantal rings, and to [2], [3], [4] and [5] for structural results of algebras generated by minors.

Let us consider the matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}$$

where the x_{ij}'s are indeterminates over a field $\Bbbk$. With $[ij] = x_{1i}x_{2j} - x_{1j}x_{2i}$, one has

$$[12][34] - [13][24] + [14][23] = 0.$$

This is the Plücker relation, and it is the only minimal relation in the sense that it generates the ideal of relations. In fact, the case $t = \min\{m,n\}$ is well understood in general, even if anything but trivial: If $t = \min\{m,n\}$ the Plücker relations generate the ideal of relations between the t-minors of X. In particular, there are only quadratic minimal relations.

This changes already for 2-minors of a 3×4-matrix. To identify a minor we have now to specify rows and columns indices. Denote by $[ij|pq]$ the minor of X with row indices i, j and column indices p, q. Of course, the Plücker relations are still present, but they are

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no more sufficient. Cubics appear among the minimal relations, for example

$$\det\begin{pmatrix} [12|12] & [12|13] & [12|14] \\ [13|12] & [13|13] & [13|14] \\ [23|12] & [23|13] & [23|14] \end{pmatrix} = 0; \tag{0.1}$$

see [2].

One reason why the case of maximal minors is easier than the general case emerges from a representation-theoretic point of view. Let $\Bbbk$ be a field of characteristic 0, A_t denote the subalgebra of the polynomial ring $\Bbbk[X] = \Bbbk[x_{ij}]$ generated by the t-minors of X. When $t = m \leq n$ the ring A_t is the coordinate ring of the Grassmannian $G(m,n)$ of all m-dimensional subspaces of a vector space W of dimension n. In the general case, A_t is the coordinate ring of the Zariski closure of the image of the following morphism of affine spaces:

$$\Lambda_t \;:\; \mathrm{Hom}_{\Bbbk}(W,V) \to \mathrm{Hom}_{\Bbbk}(\bigwedge^t W, \bigwedge^t V), \;\; \Lambda_t(\phi) = \wedge^t \phi,$$

where V is a vector space of dimension m. Notice that the group $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$ acts on each graded component $(A_t)_d$ of A_t. If $t = \min\{m,n\}$, then each $(A_t)_d$ is actually an *irreducible* G-representation. This is far from being true in the general case, and this complicates the situation tremendously.

In this paper we will exhibit quadratic and cubic minimal relations between t-minors, that naturally appear in a $m \times n$-matrix for $t \geq 2$. The action of G on A_t induces a G-action also on the ideal of relations J_t. Therefore it suffices to describe the highest weight vectors of the G-irreducible subrepresentations of J_t.

Each relation f between minors gives rise to a mirror relation denoted by f', namely the one obtained by switching columns and rows.

The quadratic relations will be completely described in Subsection 2.1 in terms of the irreducible G-representations associated to them and their highest weight vectors: we call the latter $\mathbf{f}_{u,v}$ where u and v vary in $\{0,\dots,t\}$ and are such that $u+v$ is even and $u \neq v$, see (2.2). These correspond to Plücker relations if and only if $u = 0$ or $v = 0$. So, if $t \geq 3$, Plücker relations are not the only quadratic relations. By construction one has $\mathbf{f}'_{u,v} = \mathbf{f}_{v,u}$.

As (0.1) shows, minimal cubic relations exist already for $t = 2$. We will see that, every time t increases by 1, a new type of minimal cubic relation comes up. We give the corresponding irreducible G-representations and highest weight vectors in Subsections 2.2 and 2.4. For a given t the cubic relations we describe are of two kinds (up to mirror), even and odd. We denote their highest weight vector by $\mathbf{g}_u$, see (2.7), with $1 \leq u \leq \lfloor t/2 \rfloor$ for the even relations and by $\mathbf{h}_u$, see (2.10), with $2 \leq u \leq \lceil t/2 \rceil$ for the odd. In Subsection 2.5 we will describe how one can find the especially appealing determinantal relations, not necessarily minimal, like (0.1).

We can prove that there are no further minimal cubic relations only for $t = 2$ and $t = 3$ (Subsections 3.3 and 3.4). Nevertheless we conjecture that the highest weight relations we have identified generate the ideal of relations for all t,m,n (Conjecture 2.12).

In Section 3 we have collected the evidence supporting our conjecture. To a large extent it is based on computer calculations involving various tools like Singular [10] and Lie [14] and algorithms developed by the authors. Using the toric deformation of [3], we first determine the Castelnuovo-Mumford regularity of A_t in Theorem 3.1 for all t,m,n. In conjunction with a priori information on the Hilbert function of A_t, it provides degree bounds for Gröbner basis calculations by which we have verified the conjecture in case $t=2$ for $m,n \le 5$ and $m=4$, n arbitrary, as documented in Subsection 3.2. (A duality argument, see Proposition 1.3, then implies it for $t=3$, $m=n=5$.) The result for $4\times n$ matrices is based on (the easy) Theorem 3.4 by which minimal relations of t-minors of a $m\times n$ matrix have already to “live” in an $m\times(m+t)$ matrix.

By computations based on Young symmetrizers we can exclude that there exist degree 4 minimal relations for $t=2$, and this may be the strongest argument for the conjecture. (With more effort, these computations could be pushed until degree 6.) In the last two subsections 3.5 and 3.6 we show that we have found all relations that exist for “very strong” combinatorial reasons. At least, they make it very unlikely that our relations are incomplete in degree 3.

To indicate our main method of proof we have to specify some technical details. In representation theoretic terms, A_t is the subalgebra of $\Bbbk[X]$ generated by the unique copy of the irreducible G-representation $\bigwedge^t V\otimes\bigwedge^t W^*$ in $\Bbbk[X]$. By the universal property of the symmetric algebra one has a presentation

$$A_t = \operatorname{Sym}(E\otimes F^*)/J_t, \qquad E=\bigwedge^t V,\ F=\bigwedge^t W.$$

The problem we discuss is to describe a (minimal) system of generators of J_t as a (G-)ideal in $S_t=\operatorname{Sym}(E\otimes F^*)$. It is one of the two main obstructions to the solution of the problem that the decomposition of S_t into G-irreducibles is not known. (In fact, to know it is equivalent to knowing the $\mathrm{GL}(V)$-decomposition of $L_\mu(\bigwedge^t V)$ for all partitions μ, a completely open plethysm problem.) Fortunately, by the work of De Concini, Eisenbud and Procesi [8], from the decomposition of A_t one can link the decompositions of S_t and J_t easily.

In order to describe minimal relations we develop combinatorial techniques to identify irreducible representations in J_t and to decide whether they are in the span of lower degree representations.

At this point it is inevitable to work simultaneously with the larger group $H=\mathrm{GL}(E)\times\mathrm{GL}(F)$, despite the fact that J_t is not an H-ideal. After the introduction of some notation and of our objects in Subsections 1.1 and 1.2, we develop the representation theoretic structure of S_t in Subsections 1.3 and 1.5.

The intermediate subsection 1.4 is devoted to a formula that will allow us to derive relations with prescribed G-type from lower degree relations. Lemma 1.11, which may be of interest beyond our application, helps us in specific cases to overcome the second main obstruction, namely the lack of understanding the relationship between the algebra structure of S_t and its G-structure. In contrast, the H-structure is well understood by [8], and we can combine it with Pieri's formula in order to (dis)prove that certain representations in J_t are minimal.

It turns out that all the minimal relations we have found exist for “shape reasons” encoded in the G-decompositions of the modules $L_\lambda E\otimes L_\lambda F^*$ and Pieri's formula. Indeed, it is our feeling, mainly based on computational experience, that these are, roughly speaking, the only reasons for a irreducible G-representation to give a minimal relation. The feeling just expressed is made more precise in Conjecture 3.8.

In view of the representation theoretic approach we will assume throughout that the base field $\Bbbk$ has characteristic 0.

1. The representation theoretic structure

Representation theory will guide us in our search for relations between the t-minors, in proving existence and proving non-existence. Before starting, we need to introduce some notation.

1.1. **Notation.** Let $\Bbbk$ be a field of characteristic 0, V a $\Bbbk$-vector space of dimension n and E a finite dimensional rational $\mathrm{GL}(V)$-representation (or $\mathrm{GL}(V)$-module). Then E can be decomposed in irreducible $\mathrm{GL}(V)$-modules, which are parametrized by partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ and $\lambda_1 \leq n$. More precisely, E can be written as a direct sum of *Schur modules* $L_\lambda V$ and of their duals. Since there is a $\mathrm{GL}(V)$-equivariant isomorphism $(L_\lambda V)^* \cong L_\lambda(V^*)$, there is no danger in writing $L_\lambda V^*$ for $(L_\lambda V)^*$, and from now on we will do it. We follow the notation of Weyman [18], so $L_{(1,1,\dots,1)}V \cong \mathrm{Sym}^d V$ and $L_{(d)}V \cong \bigwedge^d V$. (Fulton and Harris [12] use the dual convention). We will write $\lambda \vdash d$ if $\lambda_1 + \cdots + \lambda_k = d$. It might be that we will write a partition grouping the equal terms together: For example we may write $(7^3, 2, 1^2)$ for $(7,7,7,2,1,1)$. We can feature a partition λ as a *(Young) diagram* (sometimes we will refer to it also as a *shape*), that we will still denote by λ, namely:

$$\lambda = \{(i,j) \in \mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\} \ : \ i \leq k \text{ and } j \leq \lambda_i\}.$$

It is convenient to think at a diagram as a sequence of rows of boxes, for instance the diagram associated to the partition $\lambda = (6,5,5,3,1)$ features as

$$\lambda \ = \ \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ & \ \\ \cline{1-5} \ & \ & \ & \ & \ \\ \cline{1-5} \ & \ & \ \\ \cline{1-3} \ \\ \cline{1-1} \end{array}$$

Given a diagram λ, a *(Young) tableau* Λ of shape λ on $\{1, \dots, r\}$ is a filling of the boxes of λ by letters in the alphabet $\{1, \dots, r\}$. For instance, the following is a tableau of shape $(6,5,5,3,1)$ on $\{1, \dots, r\}$, provided $r \geq 7$.

$$\Lambda \ = \ \begin{array}{|c|c|c|c|c|c|}\hline 3 & 5 & 4 & 3 & 2 & 7 \\ \hline 2 & 1 & 7 & 6 & 4 \\ \cline{1-5} 2 & 2 & 3 & 1 & 2 \\ \cline{1-5} 5 & 6 & 7 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array}$$

Formally, a tableau Λ of shape λ on $\{1, \dots, r\}$ is a map $\Lambda : \lambda \to \{1, \dots, r\}$. The *content* of Λ is the vector $c(\Lambda) = (c(\Lambda)_1, \dots, c(\Lambda)_r) \in \mathbb{N}^r$ such that $c(\Lambda)_p = |\{(i,j) \ : \ \Lambda(i,j) = p\}|$. A tableau is *standard* if it is rows increasing and columns nondecreasing. It turns out that, once a basis of V has been fixed, let us say $e_1,\dots,e_n$, the set of standard tableaux of shape λ on $\{1,\dots,n\}$ is in one-to-one correspondence with a basis of $L_\lambda V$. Moreover, we can identify $\mathrm{GL}(V)$ with the group of invertible $n\times n$-matrices with entries in $\Bbbk$: A matrix $A\in\mathrm{GL}(V)$ acts on V by multiplication on the left of the column vectors.

Let us recall the following explicit construction of a Schur module. Let $\lambda\vdash d$ be a diagram and Λ be a tableau of shape λ such that $c(\Lambda)=(1,1,\dots,1)\in\mathbb{N}^d$. Let Σ_d be the symmetric group on d elements, and let us define the following subsets of it:

$$\mathscr{C}_\Lambda=\{\sigma\in\Sigma_d\ :\ \sigma \text{ preserves each column of } \Lambda\},$$
$$\mathscr{R}_\Lambda=\{\tau\in\Sigma_d\ :\ \tau \text{ preserves each row of } \Lambda\}.$$

The symmetric group Σ_d acts on $\bigotimes^d V$ by

$$\sigma(v_1\otimes\cdots\otimes v_d)=v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(d)},\quad \sigma\in\Sigma_d,\ v_i\in V.$$

and extending $\Bbbk$-linearly. With these notation, the *Young symmetrizer* (with respect to Λ) is the following map:

$$\begin{aligned}\mathbb{Y}_\Lambda:\bigotimes^d V&\to\bigotimes^d V\\ v_1\otimes\cdots\otimes v_d&\mapsto\sum_{\sigma\in\mathscr{C}_\Lambda}\sum_{\tau\in\mathscr{R}_\Lambda}(-1)^\tau\sigma\tau(v_1\otimes\cdots\otimes v_d).\end{aligned}$$

It turns out that there is a $\mathrm{GL}(V)$-isomorphism $\mathbb{Y}_\Lambda(\bigotimes^d V)\cong L_\lambda V$. For a tableau Γ of shape λ on $\{1,\dots,n\}$ we set

$$\mathbb{Y}_\Lambda(\Gamma)=\mathbb{Y}_\Lambda(e_{\Gamma(1,1)}\otimes\cdots\otimes e_{\Gamma(1,\lambda_1)}\otimes\cdots\otimes e_{\Gamma(k,1)}\otimes\cdots\otimes e_{\Gamma(k,\lambda_k)}).$$

Notice that $\mathbb{Y}_\Lambda$ is alternating in the rows of λ: if Γ' arises from Γ by the exchange of two entries in the same row, then

$$\mathbb{Y}_\Lambda(\Gamma)=-\mathbb{Y}_\Lambda(\Gamma').$$

In literature, the Young symmetrizers are often defined by letting first act the column-preserving permutations and then the row-preserving ones. Such a definition does not yield an alternating map. However, the two definitions lead to the same theory, as explained in the book of Procesi [17, Section 9.2].

We recall that an irreducible rational $\mathrm{GL}(V)$-representation $F\subseteq E$ can be identified by its highest weight. We fix a basis of V so that we can speak of diagonal or triangular matrices in $\mathrm{GL}(V)$. A *weight vector* of E of *weight* $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n$ is a vector $v\in E$ such that $\operatorname{diag}(\mathbf{a})v=a_1^{\alpha_1}\cdots a_n^{\alpha_n}v$, where $\operatorname{diag}(\mathbf{a})$ is an arbitrary diagonal matrix in $\mathrm{GL}(V)$ with diagonal $a_1,\dots,a_n\in\Bbbk$. The *highest weight* of F is the lexicographically largest weight of a weight vector of F, and the corresponding weight vector v, unique up to scalar, is called a *highest weight vector*. The highest weight is independent of the basis chosen in V and represents the irreducible representation up to isomorphism. If E is polynomial, then $F\cong L_\lambda V$ if and only if ${}^{\mathrm{t}}\lambda$ is the weight of v. (We remind the reader that ${}^{\mathrm{t}}\lambda$ is the transpose partition of λ, given by ${}^{\mathrm{t}}\lambda_i=|\{j:\lambda_j\geq i\}|$.)

Let $\mathrm{U}_-(V)\subseteq\mathrm{GL}(V)$ be the subgroup of lower triangular matrices with 1's on the diagonal. Then a $\mathrm{U}_-(V)$-invariant vector v of a rational representation E is the highest weight vector of an irreducible $\mathrm{GL}(V)$-module $F\subseteq E$.

Given the $\mathrm{GL}(V)$-module E, we define

$$E_\lambda$$

to be the sum of all its irreducible $\mathrm{GL}(V)$-submodules that are isomorphic to $L_\lambda V$. Then $E_\lambda \cong (L_\lambda V)^m$ for some integer $m \geq 0$. We denote the *multiplicity* m *of* λ *in* E by

$$\mathrm{mult}_\lambda(E).$$

If $\mathrm{mult}_\lambda(E) \leq 1$ for all λ, then E is called *multiplicity free*. If $\mathrm{mult}_\lambda(E) > 0$, we will say that λ *occurs in* E.

We will mainly be concerned with representations of the group $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$ for vector spaces V and W. Up to isomorphism its irreducible polynomial representations are the modules $L_\gamma V \otimes L_\lambda W$. Actually, we will deal especially with the rational irreducible G-modules $L_\gamma V \otimes L_\lambda W^*$. The notation just introduced will be applied analogously to pairs $(\gamma|\lambda)$. So we will speak of *bi-diagrams* $(\gamma|\lambda)$, *bi-tableaux* etc. We have also to speak about *bi-weights* and *bi-weight vectors*. The *highest bi-weight vector* of $L_\gamma V \otimes L_\lambda W^*$ is the (unique up to scalar) U-invariant element of $L_\gamma V \otimes L_\lambda W^*$, where $U = \mathrm{U}_-(V) \times \mathrm{U}_+(W)$: equivalently, it is the element of bi-weight $(({}^{\mathrm{t}}\gamma_1, \ldots, {}^{\mathrm{t}}\gamma_h)|(-{}^{\mathrm{t}}\lambda_k, \ldots, -{}^{\mathrm{t}}\lambda_1))$.

1.2. **The algebras A_t and their defining ideals.** First of all, let us introduce our objects. Let $\Bbbk$ be a field of characteristic 0, m and n two positive integers such that $m \leq n$ and

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & \cdots & x_{mn} \end{pmatrix}$$

a $m \times n$ matrix of indeterminates over $\Bbbk$. Moreover let

$$R(m,n) = \Bbbk[x_{ij} \,:\, i = 1, \ldots, m,\ j = 1, \ldots, n]$$

be the polynomial ring in mn variables over $\Bbbk$. We are interested in the $\Bbbk$-subalgebra $A_t(m,n) \subseteq R(m,n)$ generated by the t-minors of the matrix X. We will use the standard notation for a t-minor, namely, given two sequences $1 \leq i_1, \ldots, i_t \leq m$ and$1 \leq j_1, \ldots, j_t \leq n$, we write

$$[i_1, \ldots, i_t | j_1, \ldots, j_t]$$

for the determinant of the $t \times t$-submatrix of X with row indices $i_1, \ldots, i_t$ and the column indices $j_1, \ldots, j_t$. So we have

$$A_t(m,n) = \Bbbk[[i_1, \ldots, i_t | j_1, \ldots, j_t] : 1 \leq i_1 < \cdots < i_t \leq m, 1 \leq j_1 < \cdots < j_t \leq n] \ \subseteq\ R(m,n).$$

When there is no danger of confusion, we will simply write R and A_t instead of, respectively, $R(m,n)$ and $A_t(m,n)$. Now let V and W be $\Bbbk$-vector spaces of dimension, respectively, m and n. Let us fix a basis $\{e_1, \ldots, e_m\}$ of V and one of W, say $\{f_1, \ldots, f_n\}$. We have a natural action of $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$ on R, namely the one induced by

$$(A,B) \cdot X = AXB^{-1} \ \ \forall\, A \in \mathrm{GL}(V), B \in \mathrm{GL}(W).$$

For $1 \leq t \leq m$ the $\Bbbk$-algebra A_t is a G-invariant subspace of R. Moreover this action respects the $\mathbb{N}$-grading of R, so, actually, any degree component R_d is a finite rational G-representation. Moreover, the decomposition of R into irreducible G-modules is available, known as the *Cauchy formula*: It is easy to show that the natural isomorphism $\mathrm{Sym}(V \otimes W^*) \cong R$ is G-equivariant, and the Cauchy formula gives the decomposition

$$R_d \cong \mathrm{Sym}^d(V \otimes W^*) \cong \bigoplus_{\lambda \vdash d} L_\lambda V \otimes L_\lambda W^*. \tag{1.1}$$

where the direct sum is extended over all the partitions λ of d such that $\lambda_1 \leq m$. The decomposition of the subrepresentation $A_t \subseteq R$ in irreducible G-modules can be deduced from the work of De Concini, Eisenbud and Procesi [8]. Before describing it, we want to point out that we will consider the graded structure on A_t such that the t-minors have degree 1, so that $(A_t)_d \subseteq R_{td}$.

Definition 1.1. A partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash e$ is called (t,d)-*admissible* if $e = td$ and $k \leq d$

We have the decomposition

$$(A_t)_d \cong \bigoplus_{\lambda \vdash td} L_\lambda V \otimes L_\lambda W^* \tag{1.2}$$

where the direct sum runs over the (t,d)-admissible partitions. See [5, 3.3] for this compact description of A_t.

To a pair of standard tableaux of shape λ on $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively, we can associate a product of minors $\Delta \in R$ of shape λ, namely $\Delta = \delta_1 \cdots \delta_k$ where δ_i is a λ_i-minor. For example:

$$\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \ , \ \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 2 \\ \cline{1-1}\end{array} \right) \rightsquigarrow \quad [1,3,4|2,3,5] \cdot [3|2]$$

As said in the introduction we want to understand the relations of the t-minors of X. Therefore we have to investigate the kernel $J_t(m,n)$ of the natural graded homomorphism

$$\pi \ : \ S_t(m,n) = \mathrm{Sym}\left(\bigwedge^t V \otimes \bigwedge^t W^* \right) \to A_t(m,n).$$

When there is no ambiguity we will just write S_t and J_t instead of $S_t(m,n)$ and $J_t(m,n)$.

Remark 1.2. Consider the following numerical situations:

$$\begin{array}{ll} \text{(a)} & t = 1 \\ \text{(b)} & n \leq t+1 \\ \text{(c)} & t = m \end{array} \tag{1.3}$$

In the cases (a) and (b) the algebra A_t is a polynomial ring, so that $J_t = 0$. In case (a) this is trivial, and in case (b) it follows from the fact that the Krull dimension of A_t is equal to mn if $t < m$ (see the book of Bruns and Vetter [7, Prop. 10.16(b)]). In the case (c) A_t is the coordinate ring of the Grassmannian $G(m,n)$. In this case the ideal J_t is generated by the Plücker relations. In particular it is generated in degree 2.

Notice that the group G acts in an obvious way on the polynomial ring S_t. Furthermore the map π is G-equivariant. This implies that J_t is a G-subrepresentation of S_t, so that it has a decomposition as a direct sum of irreducible representations. Moreover, if $L_\gamma V \otimes$

$L_\lambda W^*$ is an irreducible representation of S_t, then it collapses to zero or it is mapped isomorphically to itself. So (1.2) implies that $L_\gamma V \otimes L_\lambda W^* \subseteq J_t$ whenever $\gamma \neq \lambda$. However it is difficult to say anything more at this point. In fact, a decomposition of S_t as direct sum of irreducible representations is unknown, falling into the category of *plethysm problems*.

Let us note a useful duality that does not depend on representation theory.

Proposition 1.3. *The graded algebras $A_t(n,n)$ and $A_{n-t}(n,n)$ are isomorphic.*

Proof. We use the notation of [7, Section 4]. In the coordinate ring $G(Y)$ of the coordinate ring of the Grassmannian $G(n,2n)$ we consider the subalgebra P generated by all n-minors with exactly t columns in the first n columns of the matrix Y. The standard homomorphism ϕ that maps $G(Y)$ to $\Bbbk[Z]$ where Z is an $n \times n$-matrix of indeterminates, maps P surjectively onto $A_t(n,n)$. However, $\phi\mid_P$ is an isomorphism since the kernel of ϕ is generated by $\Delta \pm 1$ where Δ is the minor $[n+1,\dots,2n]$ of Y. As $\phi\mid_P$ is a homomorphism of graded algebras, its kernel is generated by homogeneous elements, but $\Delta \pm 1$ has no homogeneous nonzero multiples.

If we consider dehomogenization with respect to the minor $[1,\dots,n]$ we obtain an isomorphism of P and $A_{n-t}(n,n)$. □

A special case of the proposition is the isomorphism of $A_{n-1}(n,n)$ and $A_1(n,n)$ observed above.

In the following we will often speak about "minimal generators" or even "minimal subspaces" of J_t. Let us make this terminology precise. An element x in J_t is a *minimal generator* if its image under the natural map $J_t \to J_t/(S_t)_1 \cdot J_t$ is non-zero, and $x_1,\dots,x_n$ are said to be *minimal generators* if their images in $J_t/(S_t)_1 \cdot J_t$ are $\Bbbk$-linearly independent, in other words, if $x_1,\dots,x_n$ can be extended to a minimal system of generators. A $\Bbbk$-subspace Q is *minimal* if the natural map $Q \to J_t/(S_t)_1 \cdot J_t$ is injective.

It should be noted that minimal relations of t-minors stay minimal if the matrix is increased and can be extended to minimal relations of t'-minors for $t' \geq t$. In fact, in [5, 5.2] the following has been proved:

Proposition 1.4. *$A_t(m,n)$ is a graded $\Bbbk$-algebra retract of $A_{t'}(m',n')$ if $n'-n, m'-m \geq t'-t$.*

1.3. **The passage to the tensor algebra.** In order to avoid the difficulties just described, we go "one more step to the left", in a way that we are going to outline.

Consider the Segre product $T_t(m,n)$ of the tensor algebras $T(\bigwedge^t V)$ and $T(\bigwedge^t W^*)$ which is (G-equivariantly isomorphic to) the tensor algebra $T(\bigwedge^t V \otimes \bigwedge^t W^*)$. We have the projection from the tensor algebra to the symmetric algebra

$$\phi : T_t(m,n) \to S_t(m,n).$$

whose kernel is a two-sided ideal generated in degree 2. When it does not raise confusion, we simply write T_t for $T_t(m,n)$. Finally, we have a G-equivariant surjective graded homomorphism

$$\psi = \pi \circ \phi : T_t(m,n) \to A_t(m,n).$$

Its kernel is denoted by $K_t(m,n)$ or simply K_t. Since $\mathrm{Ker}(\phi)$ is generated in degree two and J_t is generated in degree at least two, in order to find the maximum degree of a minimal generator of J_t we can study the maximum degree of a minimal generator of the two-sided ideal K_t. Actually we can say more: If some element x of an irreducible subrepresentation Q of S_t is a minimal generator of J_t, then the whole $\Bbbk$-basis of such an irreducible representation consists of minimal generators of J_t. In fact, if $x \notin (S_t)_1 \cdot (J_t)_{d-1}$, then the G-equivariant map $Q \to J_t/\big((S_t)_1 \cdot (J_t)_{d-1}\big)$ has to be injective. The same holds for T_t and K_t. Therefore we are allowed to speak about "minimal irreducible representations" or "minimal bi-shapes" in the kernel.

Lemma 1.5. *Let $d \geq 3$ be an integer. An irreducible representation of $(T_t)_d$ is minimal in K_t if and only if is minimal in J_t.*

The advantages of passing to T_t are that it "separates rows and columns" (of the minors) and that its decomposition in irreducible G-representations is available, see Proposition 1.7. The disadvantage is that we have to work in a noncommutative setting. Before describing the decomposition of T_t it is convenient to introduce a definition.

Definition 1.6. We say that a diagram α is a *t-predecessor* (or simply *predecessor*) of a (t,d)-admissible diagram λ if α is $(t,d-1)$-admissible, $\alpha_1 \leq \lambda_1 \leq \alpha_1 + t$ and $\alpha_i \leq \lambda_i \leq \alpha_{i-1}$ for all $i \geq 2$.

If α is a predecessor of λ, then λ is a *successor* of α.

The notion of predecessor (or successor) reflects *Pieri's formula* (for example, see [18, Corollary 2.3.5]):

$$L_\alpha V \otimes \bigwedge^t V \cong \bigoplus L_\lambda V \tag{1.4}$$

where λ runs through the successors of α.

Proposition 1.7. *As a G-representation, $(T_t)_d$ decomposes as*

$$(T_t)_d \cong \bigoplus_{\gamma,\lambda} (L_\gamma V \otimes L_\lambda W^*)^{n(\gamma,\lambda)},$$

where the sum runs over the (t,d)-admissible diagrams γ and λ with $\gamma_1 \leq m$, $\lambda_1 \leq n$; the multiplicity $n(\gamma,\lambda) = \operatorname{mult}_{(\gamma|\lambda)}(T_t)$ is a positive integer, described recursively as follows:

(1) *If $\gamma = \lambda = (t)$, then $n(\gamma,\lambda) = 1$;*
(2) *If γ and λ are (t,d)-admissible partitions with $d > 1$, then $n(\gamma,\lambda) = \sum n(\alpha,\beta)$ where the sum runs over all t-bi-predecessors $(\alpha|\beta)$ of $(\gamma|\lambda)$.*

Proof. It is enough to find a decomposition of $\bigotimes^d \bigwedge^t V$ as a $\operatorname{GL}(V)$-representation and of $\bigotimes^d \bigwedge^t W^*$ as a $\operatorname{GL}(W)$-representation. (As mentioned above, the irreducible G-representations in $(T_t)_d$ are all of type $L_\gamma V \otimes L_\lambda W^*$ where $L_\gamma V$ is an irreducible $\operatorname{GL}(V)$-representation in $\bigotimes^d \bigwedge^t V$ and $L_\lambda W^*$ is an irreducible $\operatorname{GL}(W)$-representation in $\bigotimes^d \bigwedge^t W^*$). Now Pieri's formula and an induction easily yield the conclusion. $\square$

While the decompositions described in (1.1) and in (1.2) are multiplicity free, the numbers $n(\gamma,\lambda)$ may be, and in fact usually are, bigger than 1. As the reader will realize in the course of the paper, this is a major obstacle to saying something about the relations between minors.

Since the decomposition of A_t is known, we can easily compare the decompositions of S_t and J_t. In Section 2 the comparison will allow us to identify certain minimal relations. The next proposition follows immediately from (1.2).

Proposition 1.8. *Let γ and λ be (t,d)-admissible partitions for some $d \geq 0$. Then*

$$\operatorname{mult}_{(\gamma|\lambda)}(J_t) = \begin{cases} \operatorname{mult}_{(\gamma|\lambda)}(S_t) & \text{if } \gamma \neq \lambda, \\ \operatorname{mult}_{(\gamma|\lambda)}(S_t) - 1 & \text{if } \gamma = \lambda. \end{cases}$$

Remark 1.9. It is worth noting that Pieri's formula completely governs the structure of the G-stable ideals in A_t.

(a) Let us first discuss the case $t = 1$. Let $R = R(m,n)$ and consider the ideal I_σ generated by $R_{(\sigma|\sigma)}$. By a theorem of [8] (also see [7, 11.15]) one has

$$I_\sigma = \bigoplus_\tau R_{(\tau|\tau)} \tag{1.5}$$

where the sum is extended over all diagrams $\tau \supseteq \sigma$.

(b) Now let λ be (t,d)-admissible, and let B_σ be the ideal in A_t generated by $(A_t)_\sigma = (A_t)_{(\sigma|\sigma)}$. Then

$$B_\sigma = A_t \cap I_\sigma = \bigoplus (A_t)_\tau \tag{1.6}$$

where the sum is taken over all partitions τ that arise as iterated t-successors of σ.

The inclusion $\subseteq$ is a direct consequence of Pieri's formula whereas the opposite inclusion follows from a theorem of Whitehead [19, Theorem 7.2] who determined the (necessarily multiplicity free) decomposition of $R_{(\sigma|\sigma)} \cdot R_{(\tau|\tau)}$ for arbitrary σ and τ, showing that the irreducibles appearing in it are exactly those that come up in the Littlewood-Richardson formula for $L_\sigma V \otimes L_\tau V$. For $\tau = (d)$ the Littlewood-Richardson formula specializes to Pieri's formula. Then (1.6) follows by induction.

1.4. **A formula for successors of a Schur module.** In order to exclude a bi-diagram $(\gamma|\lambda)$ from being minimal in J_t we must find a bi-diagram $(\gamma'|\lambda')$ such that $(\gamma|\lambda)$ occurs in $(S_t)_1 \cdot (J_t)_{(\gamma'|\lambda')}$. In this subsection we will derive a formula which allows us to explicitly build a highest weight vector of shape γ from a highest weight vector of shape γ'. The formula will be crucial for concrete computations in Section 3.

More precisely, let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash N$ and $\gamma = (\gamma_1, \dots, \gamma_h) \vdash N + t$ be two diagrams. Furthermore, let Λ and Γ be tableaux of shapes λ and γ on $\{1, \dots, N\}$ and $\{1, \dots, N+t\}$, of contents $(1, \dots, 1) \in \mathbb{N}^N$ and $(1, \dots, 1) \in \mathbb{N}^{N+t}$, respectively. We know that an isomorphic copy of $\mathbb{Y}_\Gamma(\bigotimes^{N+t} V)$ is a direct summand of $\mathbb{Y}_\Lambda(\bigotimes^N V) \otimes (\bigotimes^t V)$ if and only if $\lambda \subseteq \gamma$. However, in general $\mathbb{Y}_\Gamma(\bigotimes^{N+t} V)$ is not contained in $\mathbb{Y}_\Lambda(\bigotimes^N V) \otimes (\bigotimes^t V)$, regardless of the choice of Γ. Below, we will discuss how to produce an element in $\bigotimes^{N+t} V$ which is the highest weight vector of one of the isomorphic copies of $L_\gamma V$ contained in $\mathbb{Y}_\Lambda(\bigotimes^N V) \otimes (\bigotimes^t V)$ under the condition that γ is built from λ by adding t boxes in different columns, or, by Pieri's formula, shows up in $L_\lambda \otimes \bigwedge^t V$, and this is the case in which we are interested.

More precisely, let γ be obtained by adding the t boxes

$$(i_1, j_{1,1}), \dots, (i_1, j_{1,s_1}), (i_2, j_{2,1}), \dots, (i_2, j_{2,s_2}), \dots, (i_p, j_{p,1}), \dots, (i_p, j_{p,s_p})$$

to λ such that

(i) $1 \leq i_1 < i_2 < \cdots < i_p \leq h$;
(ii) if $q > r$, then $j_{q,a} < j_{r,b}$ for all a and b. Moreover $j_{r,a} < j_{r,b}$ whenever $a < b$.

Let us define a tableau $T_{\lambda,\pi,\gamma}$ of shape λ on $\{1,\dots,n\}$ for a permutation $\pi \in \Sigma_{\gamma_{i_1}}$ as follows:

$$T_{\lambda,\pi,\gamma}(i,j) = \begin{cases} \pi(j) & \text{if } i = i_\ell \text{ and } j > j_{\ell-1,s_{\ell-1}} = \gamma_{i_{\ell-1}}, \\ j & \text{otherwise,} \end{cases}$$

for all $i = 1,\dots,k$ and $j = 1,\dots,\lambda_i$ (with the convention that $j_{0,s_0} = 0$).

Example 1.10. Suppose we want to pass from $\lambda = (7,4,1) \vdash 12$ to $\gamma = (8,6,2) \vdash 16$. Given $\pi \in \Sigma_8$, the above tableu is:

$$T_{\lambda,\pi,\gamma} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & \pi(7) \\ \hline 1 & 2 & \pi(3) & \pi(4) \\ \cline{1-4} \pi(1) \\ \cline{1-1} \end{array}$$

Lemma 1.11. *The following element of $\bigotimes^{N+t} V$ is the highest weight vector of one of the copies of $L_\gamma V$ that appear in the decomposition of $\mathbb{Y}_\Lambda(\bigotimes^N V) \otimes (\bigotimes^t V) \subseteq \bigotimes^{N+t} V$:*

$$g_{\lambda \rightsquigarrow \gamma} = \sum_{\pi \in \Sigma_{\gamma_{i_1}}} (-1)^\pi \mathbb{Y}_\Lambda(T_{\lambda,\pi,\gamma}) \otimes \left(e_{\pi(j_{p,1})} \otimes \cdot\cdot \otimes e_{\pi(j_{p,s_p})} \otimes \cdots\cdots \otimes e_{\pi(j_{1,1})} \otimes \cdot\cdot \otimes e_{\pi(j_{1,s_1})}\right) \quad (1.7)$$

Proof. The element $g_{\lambda \rightsquigarrow \gamma} \in \bigotimes^{N+t} V$ belongs to $\mathbb{Y}_\Lambda(\bigotimes^N V) \otimes (\bigotimes^t V)$ by construction. Furthermore its weight is ${}^t\gamma$. Therefore, we need just to show that $g_{\lambda \rightsquigarrow \gamma}$ is $\mathrm{U}_-(V)$-invariant. Notice that a system of generators of the group $\mathrm{U}_-(V)$ is provided by the elementary transformations E_{ij}^x with $n \geq i > j \geq 1$ and $x \in \Bbbk$, acting on V via

$$E_{ij}^x(e_k) = e_k + \delta_{ik} x e_j, \qquad k = 1,\dots,n$$

(δ_{ik} is Kronecker's delta). Therefore, we need to show that $E_{ij}^x g_{\lambda \rightsquigarrow \gamma} = g_{\lambda \rightsquigarrow \gamma}$ for all $n \geq i > j \geq 1$ and $x \in \Bbbk$. Because $\mathbb{Y}_\Lambda$ is alternating on the rows, we have

$$E_{ij}^x g_{\lambda \rightsquigarrow \gamma} = g_{\lambda \rightsquigarrow \gamma} + x \sum_{\pi \in \Sigma_{\gamma_{i_1}}} (-1)^\pi g_{\lambda,\pi,\gamma}(i \mapsto j)$$

where $g_{\lambda,\pi,\gamma}(i \mapsto j)$ means $\mathbb{Y}_\Lambda(T_{\lambda,\pi,\gamma}) \otimes (e_{j_{p,1}} \otimes \cdot\cdot \otimes e_{j_{p,s_p}} \otimes \cdots\cdot \otimes e_{j_{1,1}} \otimes \cdot\cdot \otimes e_{j_{1,s_1}})$ with the unique permuted e_i replaced by e_j. Now, for all $\pi \in \Sigma_{\gamma_{i_1}}$, set $\pi' = (i\ j) \cdot \pi$. Clearly we have $g_{\lambda,\pi,\gamma}(i \mapsto j) = g_{\lambda,\pi',\gamma}(i \mapsto j)$. Moreover $(-1)^{\pi'} = -(-1)^\pi$. This implies that

$$\sum_{\pi \in \Sigma_{\gamma_{i_1}}} (-1)^\pi g_{\lambda,\pi,\gamma}(i \mapsto j) = 0,$$

so $E_{ij}^x g_{\lambda \rightsquigarrow \gamma} = g_{\lambda \rightsquigarrow \gamma}$.

It just remains to be shown that $g_{\lambda \rightsquigarrow \gamma} \neq 0$. To this goal we rewrite $g_{\lambda \rightsquigarrow \gamma}$ as

$$\sum_{T,\underline{i}} a_{T,\underline{i}} \mathbb{Y}_\Lambda(T) \otimes e_{i_1} \otimes \cdots \otimes e_{i_t},$$

where T varies among the standard tableaux of shape λ in $\{1,\dots,n\}$, $\underline{i}$ varies in $\{1,\dots,n\}^t$ and the $a_{T,\underline{i}} \in \Bbbk$ are the coefficients. Since the above representation is a linear combination of elements of a basis of $\mathbb{Y}_\Lambda(\bigotimes^N V)\otimes(\bigotimes^t V)$, it is enough to show that at least one of the $a_{T,\underline{i}}$ is not 0. This follows immediately from the fact that $\mathbb{Y}_\Lambda$ is alternating on the rows: let $\underline{i_0} = (j_{p,1},\dots,j_{p,s_p},\dots,j_{1,1},\dots,j_{1,s_1})$. The only possibly nonzero coefficient $a_{T,\underline{i_0}}$ corresponds to the tableau T_0 of shape λ such that $T(i,j)=j$ for all $(i,j)\in\lambda$. We have that

$$a_{T_0,\underline{i_0}} = \sum_{\pi\in A}(-1)^\pi(-1)^\pi = |A|,$$

where $A \subseteq \Sigma_{\gamma_{i_1}}$ consists in the permutations π such that $\pi(j_{h,k}) = j_{h,k}$ for all $h=1,\dots,p$ and $k=1,\dots,s_h$ and π preserves the rows of T_0. □

Example 1.12. In the situation of Example 1.10, we have

$$g_{\lambda\rightsquigarrow\gamma} = \sum_{\pi\in\Sigma_8}\mathbb{Y}_\Lambda(T_{\lambda,\pi,\gamma})\otimes(e_{\pi(2)}\otimes e_{\pi(5)}\otimes e_{\pi(6)}\otimes e_{\pi(8)}).$$

Remark 1.13. In view of the application of Lemma 1.11 that we have in mind let us consider the natural $\mathrm{GL}(V)$-equivariant surjective map:

$$f_d : \bigotimes^{dt} V \to \bigotimes^{d}\left(\bigwedge^{t} V\right).$$

If $N = dt$ and the starting shape $\lambda = (\lambda_1,\dots,\lambda_k)\vdash dt$ in Lemma 1.11 is (t,d)-admissible, then there exists a tableau Λ of shape λ on $\{1,\dots,dt\}$ such that $c(\Lambda) = (1,\dots,1)\in\mathbb{N}^{dt}$ and

$$f_d\left(\mathbb{Y}_\Lambda\left(\bigotimes^{dt} V\right)\right) \cong L_\lambda V.$$

In this situation, one can show that $f_d(g_{\lambda\rightsquigarrow\gamma})\neq 0$ by the same method used in the proof of Lemma 1.11. In particular, $f_d(g_{\lambda\rightsquigarrow\gamma})$ is the highest weight vector of the unique copy of $L_\gamma V$ which is a direct summand of $f_d(\mathbb{Y}_\Lambda(\bigotimes^{dt} V))\otimes(\bigwedge^t V)$.

1.5. The coarse decomposition.

Set

$$E = \bigwedge^{t} V \qquad \text{and} \qquad F = \bigwedge^{t} W.$$

Instead of the group $G = \mathrm{GL}(V)\times\mathrm{GL}(W)$ one can also consider the action of the overgroup $H = \mathrm{GL}(E)\times\mathrm{GL}(F)$ on T_t and S_t. The main advantage is that the H-structure of S_t is well-understood by the Cauchy formula:

$$S_t = \bigoplus_\mu (S_t)_\mu, \qquad (S_t)_\mu = L_\mu E\otimes L_\mu F^*, \tag{1.8}$$

with the restrictions imposed on μ by the dimensions of the involved vector spaces. However, H does not act on A_t, and the ideal J_t is not an H-submodule of S_t (apart from trivial exceptions). Therefore, in order to make full use of (1.8) one would have to understand the $\mathrm{GL}(V)$-decomposition of $L_\mu E$. For example, a bi-shape $(\gamma|\lambda)$ of partitions $\gamma,\lambda\vdash dt$ has multiplicity ≥ 1 in S_t if and only if there exists a partition $\mu\vdash d$ such that $L_\gamma V$ occurs in the decomposition of $L_\mu E$, and the same holds for $L_\lambda W^*$ in $L_\mu F^*$.

In general, the $\mathrm{GL}(V)$-decomposition of $L_\lambda E$ is an unsolved plethysm. The difficulty of the problem is illustrated by the fact that copies of $L_\gamma V$ may appear in $L_\mu E$ for several μ, and that there is no equivalence relation on partitions $\gamma, \lambda \vdash dt$ by which one could decide whether $(\gamma|\lambda)$ has multiplicity ≥ 1 in S_t. In order to illustrate the problem and for the discussion of concrete examples we include plethysms for $t=2$. The tables have been computed by Lie [14]. (Despite of the below tables, even for $t=2$ the $\mathrm{GL}(V)$-modules are not multiplicity free in general.)

$\mu=(1,1,1)$	$\mu=(2,1)$	$\mu=(3)$
(6)	(5,1)	(4,1,1)
(4,2)	(4,2)	(3,3)
(2,2,2)	(3,2,1)	

TABLE 1. Plethysms for $L_\mu(\bigwedge^2 V)$, $\mu \vdash 3$

$\mu=(1,1,1,1)$	$\mu=(3,1)$	$\mu=(2,2)$	$\mu=(2,1,1)$	$\mu=(4)$
(8)	(7,1)	(6,2)	(6,1,1)	(5,1,1,1)
(6,2)	(6,2)	(5,2,1)	(5,3)	(4,3,1)
(4,4)	(5,3)	(4,4)	(5,2,1)	
(4,2,2)	(5,2,1)	(4,2,2)	(4,3,1)	
(2,2,2,2)	(4,3,1)	(3,3,1,1)	(4,2,1,1)	
	(4,2,2)		(3,3,2)	
	(3,2,2,1)			

TABLE 2. Plethysms for $L_\mu(\bigwedge^2 V)$, $\mu \vdash 4$

Remark 1.14. Despite of the fact that J_t is not an H-ideal in S_t one could hope for the next best structure with respect to the H-action, namely that J_t is the direct sum of its intersections with the H-irreducibles $(S_t)_\mu$. Clearly, if a bi-diagram $(\gamma|\lambda)$ occurs with multiplicity 1 in S_t, then the corresponding G-irreducible must be contained in (exactly) one of the $(S_t)_\mu$. However, as soon as $\mathrm{mult}_{(\gamma|\lambda)}(S_t) \geq 2$, the inclusion $(J_t)_{(\gamma|\lambda)} \subseteq \bigoplus J_t \cap (S_t)_\mu$ may fail. In fact, it fails already in the smallest possible case, namely $(4,2|4,2)$, which has multiplicity 2 in S_2 (see Table 1) and multiplicity 1 in J_2. We will discuss the computation in Subsection 3.4.

One of the few classical known plethysms is

$$\mathrm{Sym}^d\left(\bigwedge^2 V\right) = \bigoplus_{\substack{\lambda \text{ even} \\ \lambda_1 \leq m}} L_\lambda V \tag{1.9}$$

where λ is *even* if all its parts λ_i are even; see [18, p. 63]. The plethysm (1.9) has a companion for exterior powers that we will encounter later on.

The plethysm (1.9) can be used in a ring-theoretic way as follows:

Proposition 1.15. *There are natural G-equivariant projections*

$$\alpha : S_t(m,n) \to \operatorname{Sym}(E) \sharp \operatorname{Sym}(F^*),$$
$$\beta : S_t(m,n) \to \bigwedge(E) \sharp \bigwedge(F^*),$$

where $\sharp$ denotes the Segre product.

Proof. By the universal property of the symmetric algebra, the natural homomorphisms

$$\bigotimes(E \otimes F^*) = \bigotimes E \sharp \bigotimes F^* \to \operatorname{Sym} E \sharp \operatorname{Sym} F^*,$$
$$\bigotimes(E \otimes F^*) = \bigotimes E \sharp \bigotimes F^* \to \bigwedge E \sharp \bigwedge F^*$$

are G-equivariant and factor through S_t. (Note that the Segre product of the exterior algebras is commutative.) $\square$

Now we formulate a very useful rule that simplifies many discussions. It is the representation-theoretic analogue of Proposition 1.4.

Proposition 1.16. *Let μ be a partition of d and consider partitions $\lambda = (\lambda_1, \dots, \lambda_k) \vdash td$ with $k \le d$ and $\tilde\lambda = (\lambda_1 + 1, \dots, \lambda_k + 1, 1, \dots, 1) \vdash dt + d$. If $\dim_{\Bbbk} V \ge \lambda_1 + 1$, then*

$$\operatorname{mult}_{\lambda}(L_\mu(\bigwedge^t V)) = \operatorname{mult}_{\tilde\lambda}(L_\mu(\bigwedge^{t+1} V))$$

Proof. Let us consider the map

$$\xi : \bigotimes^d \Big(\bigwedge^t V\Big) \to \bigotimes^d \Big(\bigwedge^{t+1} V\Big)$$

that extends the assignment

$$(e_{a_{1,1}} \wedge \cdots \wedge e_{a_{1,t}}) \otimes \cdots \otimes (e_{a_{d,1}} \wedge \cdots \wedge e_{a_{d,t}})$$
$$\mapsto (e_1 \wedge e_{a_{1,1}+1} \wedge \cdots \wedge e_{a_{1,t}+1}) \otimes \cdots \otimes (e_1 \wedge e_{a_{d,1}+1} \wedge \cdots \wedge e_{a_{d,t}+1})$$

$\Bbbk$-linearly; here $a_{i,j} \in \{1, \dots, \dim_{\Bbbk} V\}$, and we use the convention that $e_q = 0$ if $q > \dim_{\Bbbk} V$.

Since $\dim_{\Bbbk} V \ge \lambda_1 + 1$ the vector space Q of the $\mathrm{U}_-(V)$-invariants of weight ${}^{\mathrm{t}}\lambda$ in $\bigotimes^d \bigwedge^t V$ is contained in the subspace $\bigotimes^d \bigwedge^t V'$ where V' is generated by $e_1, \dots, e_{n-1}$, $n = \dim_{\Bbbk} V$. On this subspace ξ is injective. On the other hand, the subspace of the $\mathrm{U}_-(V)$-invariants of weight ${}^{\mathrm{t}}\tilde\lambda$ in $\bigotimes^d \bigwedge^{t+1} V$ is contained in $\xi(Q)$ since each tensor factor of each summand in the representation of such a $\mathrm{U}_-(V)$-invariant in the natural basis starts with e_1. $\square$

Definition 1.17. If a partition $\tilde\lambda$ arises from λ by prefixing λ with columns of length d, then $\tilde\lambda$ is called a *trivial extension* of λ.

Iterated application of Proposition 1.16 shows that it holds for trivial extensions in general.

For the analysis of degree 3 relations the following proposition will turn out useful.

Proposition 1.18. *Let λ be a $(t,3)$-admissible diagram with more than one predecessor. If $\dim_{\Bbbk} V \ge \lambda_1$, then $L_\lambda V$ is a direct summand of $L_{(2,1)}E$.*

Proof. By Proposition 1.16 we can assume $\lambda = (\lambda_1, \lambda_2)$. Then λ has more than one predecessor if and only if $\lambda_1 > \lambda_2 > 0$.

If $\lambda_2 \leq t$, then $(2t)$ is a predecessor of λ. Using Lemma 1.11 and Remark 1.13, we know that the element $g = g_{(2t) \rightsquigarrow \lambda}$ is the $\mathrm{U}_-(V)$-invariant of the unique copy of $L_\lambda V$ contained in $L_{(2t)} V \otimes E$. Now, $L_{(2t)} V \otimes E$ is contained in $\mathrm{Sym}^3 E \oplus L_{(2,1)} E$ or in $\bigwedge^3 E \oplus L_{(2,1)} E$, depending on the parity of t. In any case, the element g is neither symmetric nor alternating. To see this, we need to consider the ℓ monomials in the support of g:

$$(e_{a_1^i} \wedge \cdots \wedge e_{a_t^i}) \otimes (e_{b_1^i} \wedge \cdots \wedge e_{b_t^i}) \otimes (e_{c_1^i} \wedge \cdots \wedge e_{c_t^i}), \quad i = 1, \ldots, \ell.$$

Then, for all $i \in \{1, \ldots, \ell\}$, we have $1 \in \{c_1^i, \ldots, c_t^i\}$, whereas 1 does not belong to the intersection $\{a_1^i, \ldots, a_t^i\} \cap \{b_1^i, \ldots, b_t^i\}$. So $g = f + h$ with $h \in L_{(2,1)} E$ different from 0 and $f \in \mathrm{Sym}^3 E$ or $f \in \bigwedge^3 E$, depending on the parity of p. In any case, h is a $\mathrm{U}_-(V)$-invariant of weight ${}^t\lambda$, thus the $\mathrm{GL}(V)$-space generated by it, which obviously is contained in $L_{(2,1)} E$, is isomorphic to $L_\lambda V$.

If $\lambda_2 > t$, then we consider the predecessor $(\lambda_1, \lambda_2 - t)$ of λ. The proof of this case is analog to the previous one, so we do not repeat it. Let us just say that this time we show that $g_{(\lambda_1, \lambda_2 - t) \rightsquigarrow \lambda}$ is neither symmetric nor alternating by using that, for all i, $\lambda_1 \notin \{c_1^i, \ldots, c_t^i\}$ and $\lambda_1 \in \{a_1^i, \ldots, a_t^i\} \cup \{b_1^i, \ldots, b_t^i\}$. □

We introduce a class of partitions that seem to be crucial for the analysis of J_t.

Definition 1.19. We say that a partition $\lambda \vdash dt$ is of *single* $\bigwedge^t$*-type* μ if $\mu \vdash d$ is the only partition such that the $\mathrm{GL}(V)$-irreducible $L_\lambda V$ occurs in the $\mathrm{GL}(E)$-irreducible $L_\mu E$ and, moreover, has multiplicity 1 in it.

A bi-diagram $(\gamma|\lambda)$ is of *single* $\bigwedge^t$*-type* if both γ and λ are of single $\bigwedge^t$-type.

Clearly, bi-diagrams of single $\bigwedge^t$-type have multiplicity 1 in S_t (if they occur at all), but the converse does not hold, as shown by $(4,3,1|6,2)$ for $t = 2$, $d = 4$.

Remark 1.20. For every partition $\mu \vdash d$ there exists at least one partition $\lambda \vdash dt$ of single $\bigwedge^t$-type μ: just take λ to be the trivial extension of μ by prefixing it with $t-1$ columns of length d. One can use λ as an indicator for μ: a partition γ appears in μ if and only $(\gamma|\lambda)$ occurs in S_t (with the same multiplicity). Therefore the $\mathrm{GL}(V)$-decomposition of $L_\mu E$ can be reconstructed for all μ from the decomposition of S_t.

In general there exist more than one partition of single $\bigwedge^t$-type μ. The reader may check that the following (t,d)-admissible diagrams λ are of single $\bigwedge^t$-type: (i) $\lambda_1 \leq t+1$, (ii) λ is a hook, i.e. $\lambda_2 \leq 1$. By trivial extension one can construct further singe $\bigwedge^t$-type diagrams from (ii). Two other types will be encountered in Theorem 2.7 and Corollary 3.11.

Proposition 1.21. *$\lambda \vdash dt$ is of single $\bigwedge^t$-type if and only if the bi-shape $(\lambda|\lambda)$ has multiplicity* 1 *in S_t or, equivalently, does not occur in J_t.*

This follows immediately from (1.1). Single $\bigwedge^t$-type can be characterized recursively:

Proposition 1.22. *Let $\lambda \vdash dt$ and $\mu \vdash d$ be partitions such that λ occurs in $L_\mu E$. Then the following are equivalent:*

(i) *λ is of single $\bigwedge^t$-type;*

(ii) *the multiplicities of λ and of μ in $\bigotimes^d(\bigwedge^t V)$ coincide;*
(iii) *every t-predecessor λ' of λ is of single $\bigwedge^t$-type μ' where μ' is a 1-predecessor of μ, and no two distinct t-predecessors of λ share the same 1-predecessor μ' of μ.*

The proof uses only the recursive formula for multiplicities in Proposition 1.7.

In the next theorem we exploit Pieri's formula (1.4) for G and H and the Cauchy formula (1.8) simultaneously.

Theorem 1.23. (i) *Let $\mu \vdash d$ be a partition, and let M be the set of 1-successors of μ. Then the linear map*

$$(S_t)_1 \otimes (S_t)_\mu \to \bigoplus_{\nu \in M} (S_t)_\nu$$

induced by multiplication in S_t is surjective.
(ii) *Let γ and λ be (t,d)-admissible partitions. If $(\gamma|\lambda)$ occurs in $(S_t)_\mu$, but there exists a 1-predecessor μ' of μ such that all bi-predecessors of $(\gamma|\lambda)$ that occur in $(S_t)_{\mu'}$ are asymmetric, then $(\gamma|\lambda)$ is not minimal in J_t.*
(iii) *With the same notation, suppose that all bi-predecessors of $(\gamma|\lambda)$ that occur in $(S_t)_{\mu'}$ for any 1-predecessor μ' of μ are symmetric of single $\bigwedge^t$-type. Then $(\gamma|\lambda)$ is minimal in J_t.*
(iv) *Let $(\gamma|\lambda)$ be asymmetric of single $\bigwedge^t$-type μ. Then either (a) $(\gamma|\lambda)$ is not minimal in $(J_t)_\mu$ or (b) γ and λ have the same predecessors (of single $\bigwedge^t$-type).*

Proof. (i) It has already been mentioned in Remark 1.9(a) that the ideal in S_t generated by $(S_t)_\mu$ is the sum of all $(S_t)_\nu$ where ν arises from μ by the addition of boxes. This implies claim (i) (and is equivalent to it by induction).

(ii) By hypothesis all bi-predecessors of $(\gamma|\lambda)$ in $(S_t)_{\mu'}$ lie in J_t since they are asymmetric. So (i) implies that $(\gamma|\lambda)$ lies in $(S_t)_1 \cdot J_t$.

(iii) We split $(J_t)_{d-1}$ into the sum of three G-submodules, namely the sum U_1 of all G-irreducibles that are contained in $(S_t)_{\mu'}$ for 1-predecessors μ' of μ, the sum U_2 of all submodules contained in $(S_t)_{\mu'}$ for non-1-predecessors μ' of μ and a complementary summand U_3 of $U_1 \oplus U_2$ (which exists by linear reductivity of G). In general U_3 may be non-zero (see Remark 1.14), however all bi-shapes $(\gamma'|\lambda')$ in U_3 must appear in a 1-predecessor of μ as well as in a non-1-predecessor. This is impossible for single $\bigwedge^t$-type, and so $U_3 = 0$. Since $((S_t)_1 \cdot U_2) \cap (S_t)_\mu = 0$ and $U_1 \cap J_t = 0$ by hypothesis, $(\mu|\lambda)$ must indeed be minimal in J_t.

(iv) follows from (ii) and (iii). □

In particular, $(\gamma|\lambda)$ is minimal in J_t if all its bi-predecessors (with $\operatorname{mult}_{(\gamma'|\lambda')}(S_t) > 0$) are symmetric of multiplicity 1. Conversely, if all bi-predecessors are asymmetric, then $(\gamma|\lambda)$ is not minimal. However, Theorem 1.23(ii) is more precise as the following example shows: for $t = 2$ the bi-diagram $(5,3|7,1)$ belongs with multiplicity 1 only to $(S_2)_\mu$ for $\mu = (2,1,1)$. However, in $(S_2)_\nu$, $\nu = (1,1,1)$ it has no symmetric bi-predecessor, and therefore it is not minimal in J_2. (But it has the symmetric bi-predecessor $(5,1|5,1)$ of multiplicity 1 in $(S_2)_{(2,1)}$.)

On the other hand, Theorem 1.23 does not allow us to exclude that $(6,2|7,1)$ is minimal in J_2, although all relevant plethysms are known. That it is not minimal will be documented in Subsection 3.4.

Definition 1.24. The minimal relations identified in Theorem 1.23(iii) are called *shape relations*.

We do not know whether all minimal relations are shape relations. Raising this question is a main point of the paper. It is useful to introduce shape relations also in the tensor algebra:

Definition 1.25. Let $\gamma,\lambda \vdash dt$ be (t,d)-admissible. If all bi-predecessors of $(\gamma|\lambda)$ are symmetric of multiplicity 1 in T_t, then $(\gamma|\lambda)$ is called a *T-shape relation*.

Proposition 1.26. *T-shape relations are minimal in K_t, and a T-shape relation that appears in S_t is a shape relation. In particular, all T-shape relations of degree ≥ 3 are shape relations.*

Proof. The first statement follows by the same (and even simpler) arguments as for shape relations. The second is obvious, and for the third we apply Lemma 1.5. □

We will classify the T-shape relations in Subsection 3.5. However, not all shape relations are T-shape relations, as will become apparent in Subection 2.2.2.

2. Quadratic and cubic relations

In order to write down explicit polynomials representing the relations (and not just shapes or tableaux) we must introduce some notation. Let $A \subseteq \mathbb{N}$ be a set of cardinality $N < \infty$. Let us write $A = \{a_1,\dots,a_N\}$ in ascending order. Let $A_1,\dots,A_k$ be a k-partition of A: that is, $A_1 \cup \cdots \cup A_k = A$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Set $r_i = |A_i|$ and let us write $A_i = \{a_{i,1},\dots,a_{i,r_i}\}$ in ascending order. With the symbol

$$(-1)^{A_1,\dots,A_k}$$

we mean the sign of the unique permutation of A taking the sequence $a_1,\dots,a_N$ to the sequence $a_{1,1},\dots,a_{1,r_1},a_{2,1},\dots,a_{2,r_2},\dots,a_{k,1},\dots,a_{k,r_k}$. If some A_i consists of one element, so that $A_i = \{a_{i,1}\}$, we may simply write this sign as $(-1)^{A_1,\dots,A_{i-1},a_{i,1},A_{i+1},\dots,A_k}$. Given another finite set B, we will say that A is lexicographically smaller than B if $|A| < |B|$ or $|A| = |B|$ and the vector $(a_1,\dots,a_N)$ is lexicographically smaller than $(b_1,\dots,b_N)$ with $b_i \in B$ taken in ascending order. With e_A we mean $e_{a_1} \wedge e_{a_2} \wedge \cdots \wedge e_{a_N}$. Similarly for e_A^*, f_A and f_A^*. Eventually, if $B_i = \{b_{i,1},\dots,b_{i,s_i}\} \subseteq \mathbb{N}$, with the $b_{i,j}$'s taken in ascending order, are disjoint subsets for $i = 1,\dots,h$ such that $s_1 + \cdots + s_h = N$, we define the N-minor

$$[A_1,\dots,A_k|B_1,\dots,B_h] = [a_{1,1},\dots,a_{1,r_1},\dots,a_{k,1},\dots,a_{k,r_k}|b_{1,1},\dots,b_{1,s_1},\dots,b_{h,1},\dots,b_{h,s_h}]$$

In order to keep the notation transparent, we set

$$E = \bigwedge^t V \qquad \text{and} \qquad F = \bigwedge^t W$$

as in Subsection 1.5.

2.1. **Quadratic relations.** The only degree 2 (minimal) relations between 2-minors of an $m\times n$-matrix are Plücker relations, as we will see. However this is not true anymore for t-minors with $t\geq 3$. In this subsection we want to describe all the degree 2 relations between t-minors. In order to do this we need a decomposition of

$$\mathrm{Sym}^2(E\otimes F^*)$$

into irreducible G-modules. Since

$$\bigotimes^2 E=\mathrm{Sym}^2 E\oplus\bigwedge^2 E,$$

one can show (or (1.8) implies) that:

$$\mathrm{Sym}^2(E\otimes F^*)=\Big(\mathrm{Sym}^2 E\otimes\mathrm{Sym}^2 F^*\Big)\bigoplus\Big(\bigwedge^2 E\otimes\bigwedge^2 F^*\Big).$$

By Pieri's formula, we know that $\bigotimes^2 E\cong\bigoplus_{u=0}^{t}L_{\tau_u}V$, where

$$\tau_u=(t+u,t-u). \tag{2.1}$$

So the matter is just to decide whether $L_{\tau_u}V$ is in Sym^2E or in $\bigwedge^2 E$:

Lemma 2.1. *If* $\dim_{\Bbbk}V\geq 2t$, *for* $u\in\{0,\dots t\}$, *we have:*

$$L_{\tau_u}V\subseteq\mathrm{Sym}^2E\iff u \text{ is even}.$$

Proof. It is straightforward to check that the element

$$\sum_{\substack{I\cup J=\{t-u+1,\dots,t+u\}\\|I|=|J|=u}}(-1)^{I,J}(e_1\wedge e_2\wedge\cdots\wedge e_{t-u}\wedge e_I)\otimes(e_1\wedge e_2\wedge\cdots\wedge e_{t-u}\wedge e_J)$$

is a nonzero $\mathrm{U}_-(V)$-invariant. Therefore it is a highest weight vector of weight ${}^{\mathrm{t}}\tau_u=(2^{t-u},1^{2u})$. Thus it generates the irreducible $\mathrm{GL}(V)$-module $L_{\tau_u}V$. Furthermore, it is clear that $(-1)^{I,J}=(-1)^u(-1)^{J,I}$, so the claim follows. □

The same discussion holds for W^*, so Lemma 2.1 yields the desired decomposition:

$$\mathrm{Sym}^2\Big(E\otimes F^*\Big)\cong\bigoplus_{\substack{u,v\in\{0,\dots,t\}\\u+v\text{ even}}}L_{\tau_u}V\otimes L_{\tau_v}W^*.$$

Since the above decomposition is multiplicity free, exactly the asymmetric shapes belong to $(J_t)_2$:

$$(J_t)_2\cong\bigoplus_{\substack{u,v\in\{0,\dots,t\}\\u+v\text{ even}\\u\neq v}}L_{\tau_u}V\otimes L_{\tau_v}W^*.$$

So, the highest bi-weight vector of the bi-diagram $(\tau_u|\tau_v)$, with $u+v$ even and $u\neq v$, is the following element:

$$\mathbf{f}_{u,v}=\sum_{\substack{I,J\\H,K}}(-1)^{I,J}(-1)^{H,K}[1,\dots,t-u,I|1,\dots,t-v,H][1,\dots,t-u,J|1,\dots,t-v,K] \tag{2.2}$$

where the sum runs over the 2-partitions I,J of $\{t-u+1,\dots,t+u\}$ and H,K of $\{t-v+1,\dots,t+v\}$ such that $|I|=|J|=u$ and $|H|=|K|=v$. Furthermore one can assume that I is lexicographically smaller than J, so that the relation is the original one divided by 2. When we need to emphasize the size of minors, we will write $\mathbf{f}^t_{u,v}$.

Remark 2.2. Notice that $\mathbf{f}_{u,v}$ is a Plücker relation if and only if $u=0$ or $v=0$. Moreover, if $t>\max\{u,v\}$, then $\mathbf{f}^t_{u,v}$ is obtained by trivial extension from $\mathbf{f}^u_{u,v}$ or $\mathbf{f}^v_{u,v}$, according to whether $u>v$ or $v>u$ (Proposition 1.16).

2.2. **Cubic shape relations.** We will determine relations of degree 3 that are minimal generators of J_t. We will see that they are shape relations, and in Subsection 3.6 it will be shown that there are no other shape relations in degree 3.

A minimal relation between t-minors is said to be really new if it does not come from a relation between $(t-1)$-minors by trivial extension. Every time that t increases by one a really new type of minimal cubic relation shows up (provided that $m\geq\lceil t/2\rceil$ and $n\geq 2t$). Such really new cubic minimal relations exist for slightly different reasons according to whether t is even or odd, therefore we will divide this subsection in two parts.

2.2.1. *Even minimal cubics.* Despite of the title, in this first part we will construct minimal cubic relations between t-minors for any t (also for odd t). However, they will be really new only if t is even. To this purpose we define some special bi-diagrams $(\gamma_u|\lambda_u)$ for any $u=1,\dots,\lfloor t/2\rfloor$, for which both γ_u and λ_u are partitions of $3t$. In Theorem 2.4 we will prove that some of these bi-diagrams (actually each of them if the size of the matrix is big enough) are minimal irreducible representations of degree 3 in J_t.

For all $u=1,\dots,\lfloor t/2\rfloor$, we define the bi-diagram $(\gamma_u|\lambda_u)$ $(=(\gamma^t_u|\lambda^t_u)$ if we need to emphasize the size of the minors) by

$$\begin{aligned}\gamma_u&=(t+u,\ t+u,\ t-2u),\\ \lambda_u&=(t+2u,\ t-u,\ t-u).\end{aligned}\tag{2.3}$$

Notice that γ_u and λ_u are both partitions of $3t$. Furthermore, provided that $m\geq t+u$ and $n\geq t+2u$, the irreducible G-representation $L_{\gamma_u}V\otimes L_{\lambda_u}W^*$ occurs in $(T_t)_3$.

Remark 2.3. Notice that, if t is odd, the bi-diagram $(\gamma^t_u|\lambda^t_u)$ is a trivial extension of $(\gamma^{t-1}_u|\lambda^{t-1}_u)$ by Proposition 1.16. Therefore $(\gamma^t_u|\lambda^t_u)$ is really new if and only if t is even.

Theorem 2.4. *The bi-diagram $(\gamma_u|\lambda_u)$ is a T-shape relation of degree 3 and therefore a minimal irreducible representation of $J_t(m,n)$ (provided that $u\leq m-t$ and $2u\leq n-t$).*

Proof. The only bi-predecessor of $(\gamma_u|\lambda_u)$ is the bi-diagram $(\tau|\tau)$ with $\tau=(t+u,t-u)$. Since τ has degree 2, it has multiplicity 1. This shows that $(\gamma_u|\lambda_u)$ is a T-shape relation, and we can apply Proposition 1.26 □

Corollary 2.5. *The ideal J_t has some minimal generators of degree 3 apart from the cases discussed in Remark 1.2,*

Proof. In this situation the bi-diagram $(\gamma_1|\lambda_1)$ always satisfies the side condition of Theorem 2.4. □

2.2.2. *Odd minimal cubics.* Once again despite of the title, in this second part we will construct other minimal cubic relations between t-minors for any t (also for even t). However, they will be really new only if t is odd. Here the proof is more tricky than the one for the even cubics since the odd ones are not T-shape relations.

For all $u = 2, \dots, \lceil t/2 \rceil$, we define the bi-diagram $= (\rho_u|\sigma_u)$ ($(\rho_u^t|\sigma_u^t)$ if we want to emphasize the size of the minors) by

$$\begin{aligned} \rho_u &= (t+u,\ t+u-1,\ t-2u+1), \\ \sigma_u &= (t+2u-1,\ t-u+1,\ t-u). \end{aligned} \tag{2.4}$$

Notice that both ρ_u and σ_u are partitions of $3t$.

Remark 2.6. If t is even, the bi-diagram $(\rho_u^t|\sigma_u^t)$ is a trivial extension of $(\rho_u^{t-1}|\sigma_u^{t-1})$ by Proposition 1.16. So minimal relations we are going to describe now are really new only if t is odd.

Theorem 2.7. *The bi-diagram $(\rho_u|\sigma_u)$ is a shape relation (of single $\bigwedge^t$-type) and therefore a minimal irreducible representation of $J_t(m,n)$ of degree 3 (provided that $u \le m-t$ and $2u \le n-t+1$).*

Proof. Notice that ρ_u has two predecessors, namely $(t+u, t-u)$ and $(t+u-1, t-u+1)$. Also σ_u has two predecessors, namely $(t+u, t-u)$ and $(t+u-1, t-u+1)$. Therefore Proposition 1.18 implies that

$$L_{\rho_u} V \subseteq L_{(2,1)} E$$

and

$$L_{\sigma_u} W^* \subseteq L_{(2,1)} F^*.$$

So, exploiting (1.8), we get that $(\rho_u|\sigma_u)$ is a G-subrepresentation of $S_t(m,n)_{(2,1)}$. Moreover, Lemma 2.1 implies that the only two pairs (of the predecessors of ρ_u and σ_u) living in $S_t(m,n)$ are $((t+u,t-u)|(t+u,t-u))$ and $((t+u-1,t+u-1)|(t+u-1,t-u+1))$. Both of these are symmetric bi-diagrams in degree 2, and it follows that $(\rho_u|\sigma_u)$ is a shape relation. $\square$

Since $(\rho_u|\sigma_u)$ has an asymmetric bi-predecessor in T_t it is not a T-shape relation.

2.3. **A second look at the minimal relations.** The goal of this subsection is to augment the information on the minimal relations we found in this section. In Figure 1 below we will feature the bi-shapes $(\tau_u|\tau_v)$ corresponding to quadratic minimal relations when $u+v$ is even and $u < v$. Of course, one has to keep in mind that there are also the quadratic minimal relations corresponding to the mirrored bi-shapes, namely $(\tau_u|\tau_v)$ for $u > v$.

As we already noticed in Remark 2.2, Figure 1 once more shows that if $v < t$, then the relation $(\tau_u^t|\tau_v^t)$ is a trivial extension of $(\tau_u^v|\tau_v^v)$. On the other hand, the relations $(\tau_u^v|\tau_v^v)$ are really new. Therefore, whenever t increases by one, exactly $\lfloor t/2 \rfloor$ really new minimal quadratic relations appear. Furthermore, notice that the Plücker relations between t-minors are those with a $2\times t$ rectangle on one side.

Remark 2.8. We have already used the coarse decomposition

$$\operatorname{Sym}^2(E\otimes F^*) = \left(\operatorname{Sym}^2 E \otimes \operatorname{Sym}^2 F^*\right) \oplus \left(\bigwedge^2 E \otimes \bigwedge^2 F^*\right),$$

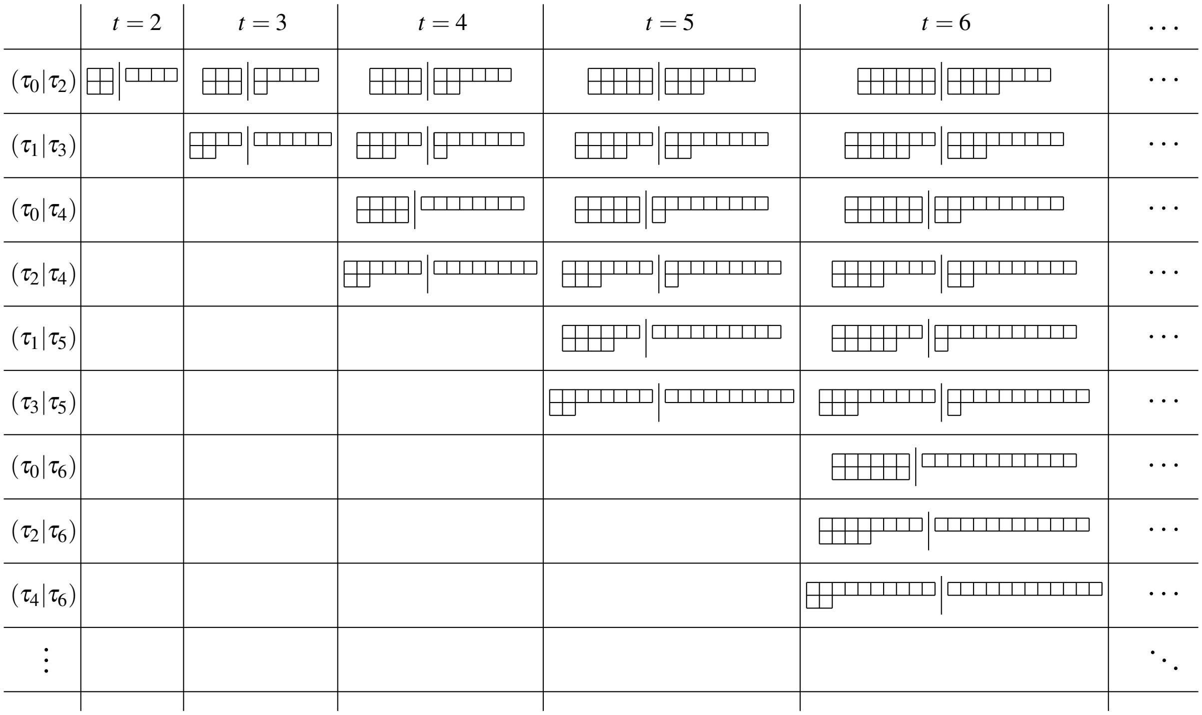


FIGURE 1. Bi-diagrams of degree 2 minimal relations

so one may wonder where the bi-diagram $(\tau_u|\tau_v)$ is placed. The answer is already clear from Subsection 2.1, namely:

(i) $(\tau_u|\tau_v)$ is in $\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 F^*$ if and only if u and v are even;
(ii) $(\tau_u|\tau_v)$ is in $\bigwedge^2 E \otimes \bigwedge^2 F^*$ if and only if u and v are odd.

Now we want to look at the shape of the found minimal cubic relations. Once again, in Figure 2 we omit the mirrored relations.

Notice that, if $t > 2u$, then the bi-shape $(\gamma_u^t|\lambda_u^t)$ is a trivial extension of $(\gamma_u^{2u}|\lambda_u^{2u})$ (Proposition 1.16). In the same vein, if $t > 2u-1$, then the bi-shape $(\rho_u^t|\sigma_u^t)$ is a trivial extension of $(\rho_u^{2u-1}|\sigma_u^{2u-1})$. In other words, every time that the size of minors t increases by 1, a new type of minimal cubic relations between t-minors comes up:

(i) If t is even, then $(\gamma_{t/2}^t|\lambda_{t/2}^t)$ starts a new series of minimal cubic relations between t'-minors, $t' \geq t$.
(ii) If t is odd, then $(\rho_{(t+1)/2}^t|\sigma_{(t+1)/2}^t)$ starts a new series of new minimal cubic relations between t'-minors, $t' \geq t$.

Remark 2.9. We have the coarse decomposition:

$$\mathrm{Sym}^3(E \otimes F^*) = \left(\mathrm{Sym}^3 E \otimes \mathrm{Sym}^3 F^*\right) \oplus \left(L_{(2,1)} E \otimes L_{(2,1)} F^*\right) \oplus \left(\bigwedge^3 E \otimes \bigwedge^3 F^*\right).$$

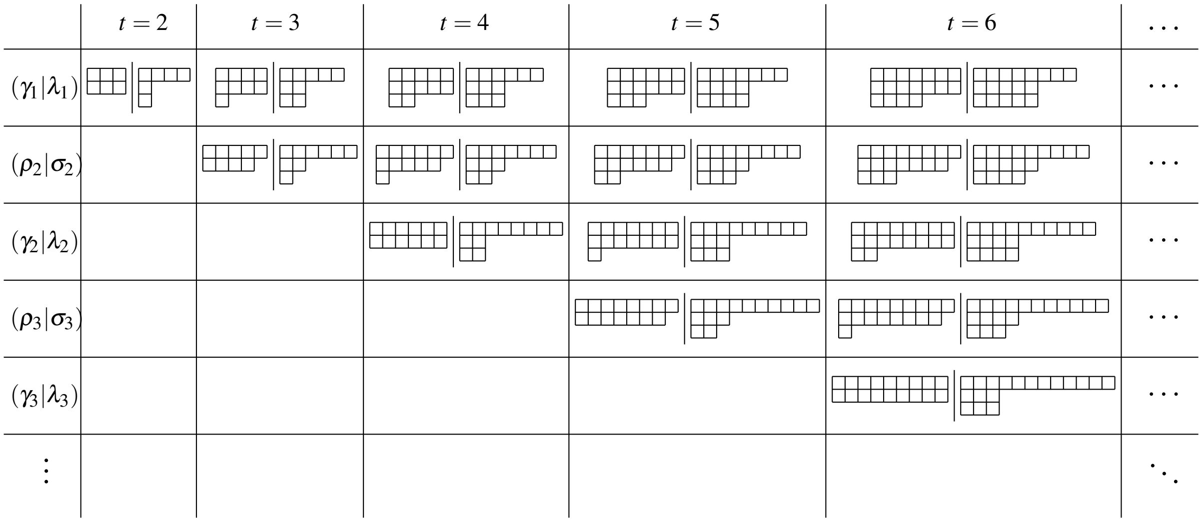


FIGURE 2. Bi-diagrams of degree 3 minimal relations

Therefore, as in Remark 2.8, we would like to place each $(\gamma_u|\lambda_u)$ and $(\rho_u|\sigma_u)$ in an irreducible H-module:

(i) $(\rho_u|\sigma_u)$ is in $L_{(2,1)}E \otimes L_{(2,1)}F^*$;
(ii) $(\gamma_u|\lambda_u)$ is in $\mathrm{Sym}^3 E \otimes \mathrm{Sym}^3 F^*$ if u is even;
(iii) $(\gamma_u|\lambda_u)$ is in $\bigwedge^3 E \otimes \bigwedge^3 F^*$ if u is odd.

For $(\rho_u|\sigma_u)$ the H-irreducible has been explicitly determined in the proof of 2.7. For each the remaining two cases one inspects the unique predecessor.

2.4. **Highest bi-weight vectors of the cubic minimal relations.** For completeness, in this subsection we will describe the polynomial corresponding to the highest bi-weight vector of any cubic relation we found up to now.

2.4.1. *Higehst bi-weight vectors of even cubics.* We need the following lemma:

Lemma 2.10. *For all $u = 1,\dots,\lfloor t/2 \rfloor$ set $K = \{t-2u+1,\dots,t+u\} \subseteq \mathbb{N}$. The highest weight vector of $L_{\gamma_u}V \subseteq \bigotimes^3(\bigwedge^t V)$ is:*

$$\sum_{A,B,C}(-1)^{A,B,C}(e_1\wedge\cdots\wedge e_{t-2u}\wedge e_{K\setminus A})\otimes(e_1\wedge\cdots\wedge e_{t-2u}\wedge e_{K\setminus B})\otimes(e_1\wedge\cdots\wedge e_{t-2u}\wedge e_{K\setminus C}) \quad (2.5)$$

where the sum runs over the 3-partitions A,B,C of K such that $|A|=|B|=|C|=u$.

Proof. Set $v=3u$ and consider a $\Bbbk$-vector space V_0 of dimension v with the $\mathrm{SL}(V_0)$-action. Let us look at

$$\bigwedge^v V_0^* \xrightarrow{\alpha} \bigotimes^v V_0^* \xrightarrow{\beta} \bigwedge^u V_0^* \otimes \bigwedge^u V_0^* \otimes \bigwedge^u V_0^* \xrightarrow{\delta} \bigwedge^{2u} V_0 \otimes \bigwedge^{2u} V_0 \otimes \bigwedge^{2u} V_0$$

Here α is antisymmetrization, namely:

$$x_1\wedge\cdots\wedge x_v \mapsto \sum_\pi (-1)^\pi x_{\pi(1)}\otimes\ldots\otimes x_{\pi(v)}.$$

In particular α is $\mathrm{SL}(V_0)$-equivariant. The map β cuts $x_1 \otimes \ldots \otimes x_v$ into blocks and maps tensor power to exterior power, so it is also $\mathrm{SL}(V_0)$-equivariant:

$$x_1 \otimes \ldots \otimes x_v \mapsto (x_1 \wedge \cdots \wedge x_u) \otimes (x_{u+1} \wedge \cdots \wedge x_{2u}) \otimes (x_{2u+1} \wedge \cdots \wedge x_v)$$

The map δ is the one that gives the isomorphism as $\mathrm{SL}(V_0)$-modules of $\bigwedge^u V_0^*$ and $\bigwedge^{2u} V_0$. It is defined, with respect to a fixed basis $e_1, \ldots, e_v$ of V_0, as follows. Let $e_1^*, \ldots, e_v^*$ be the dual basis of V_0^*. Then

$$e_{i_1}^* \wedge \cdots \wedge e_{i_u}^* = (-1)^{i_1,\ldots,i_u} e_I^* \quad \mapsto \quad (-1)^{i_1,\ldots,i_u} (-1)^{u(u-1)/2} (-1)^{I,J\setminus I} e_{J\setminus I}.$$

where $I = \{i_1, \ldots, i_u\}$ and $J = \{1, \ldots, v\}$. The constant sign $(-1)^{u(u-1)/2}$ is irrelevant for our purpose, and we will omit it. We can combine the two other signs as

$$(-1)^{i_1,\ldots,i_u} (-1)^{I,J\setminus I} = (-1)^{i_1,\ldots,i_u,J\setminus I}.$$

Now we can start from the $\mathrm{SL}(V_0)$-invariant $e_1^* \wedge \cdots \wedge e_v^* \in \bigwedge^v V_0^*$ and apply our maps. Because all the maps involved are $\mathrm{SL}(V_0)$-equivariant we end with an $\mathrm{SL}(V_0)$-invariant in $\bigotimes^3 \bigwedge^{2u} V_0$. We can assume that the permutations are increasing in the three blocks since the sign $(-1)^{i_1,\ldots,i_u}$ "corrects" the order. Thus we get

$$e_1^* \wedge \cdots \wedge e_v^* \mapsto (u!)^3 \sum_{F,G,H} (-1)^{F,G,H} (-1)^{F,G\cup H} (-1)^{G,F\cup H} (-1)^{H,F\cup G} e_{J\setminus F} \otimes e_{J\setminus G} \otimes e_{J\setminus H}.$$

where the sum is extended over all the 3-partitions F, G, H of J such that $|F| = |G| = |H| = u$. But $(-1)^{F,G\cup H} (-1)^{G,F\cup H} (-1)^{H,F\cup G}$ is constant, namely equal to $(-1)^{3u^2}$. Removing the constant sign and dividing by $(u!)^3$ yields

$$\sum_{F,G,H} (-1)^{F,G,H} e_{J\setminus F} \otimes e_{J\setminus G} \otimes e_{J\setminus H}.$$

Since the above element is $\mathrm{SL}(V_0)$-invariant, the element of $\bigotimes^3 \left(\bigwedge^t V\right)$ of the statement, namely

$$\sum_{A,B,C} (-1)^{A,B,C} (e_1 \wedge \cdots \wedge e_{t-2u} \wedge e_{K\setminus A}) \otimes (e_1 \wedge \cdots \wedge e_{t-2u} \wedge e_{K\setminus B}) \otimes (e_1 \wedge \cdots \wedge e_{t-2u} \wedge e_{K\setminus C}),$$

is $\mathrm{U}_-(V)$-invariant. Moreover, its weight is ${}^{\mathrm{t}}\gamma_u$, therefore it is the highest weight vector of $L_{\gamma_u} V$. $\square$

By a similar and simpler construction (we need not to dualize) we can compute also the highest weight vector of $L_{\lambda_u} W^* \subseteq \bigotimes^3 \left(\bigwedge^t W^*\right)$:

$$\sum_{L,M,N} (-1)^{L,M,N} (f_1^* \wedge \cdots \wedge f_{t-u}^* \wedge f_L^*) \otimes (f_1^* \wedge \cdots \wedge f_{t-u}^* \wedge f_M^*) \otimes (f_1^* \wedge \cdots \wedge f_{t-u}^* \wedge f_N^*) \quad (2.6)$$

where the sum is extended over the 3-partitions L, M, N of $\{u+1, \ldots, t+2u\}$ such that $|L| = |M| = |N| = u$.

Now we tensor the row part (2.5) and the column part (2.6) together and pass to the symmetric power $(S_t)_3$. Then each monomial appears 6 times since the monomials only depend on the set of pairs $(A, L), (B, M)$ and (C, N), but not on their order anymore. Permuting these sets does not change the sign, since both row and column factor change by the same sign. So, dividing by 6, we can assume that A, B, C is ordered lexicographically.

The element we get is the highest bi-weight vector of $L_{\gamma_u}V \otimes L_{\lambda_u}W^* \subseteq (J_t)_3$. In particular it is a minimal relation between t-minors of degree 3, and all the cubic shape relations of type $(\gamma_u|\lambda_u)$ are in the G-space generated by it. Explicitly, such a relation is:

$$\mathbf{g}_u = \sum_{\substack{A,B,C\\L,M,N}} (-1)^{A,B,C}(-1)^{L,M,N}[P,K\setminus A|Q,L][P,K\setminus B|Q,M][P,K\setminus C|Q,N], \tag{2.7}$$

where the sum runs over the 3-partitions A,B,C of $K=\{t-2u+1,\dots,t+u\}$ and L,M,N of $\{t-u+1,\dots,t+2u\}$ such that $|A|=|B|=|C|=|L|=|M|=|N|=u$ and A,B,C are ordered lexicographically. Moreover $P=\{1,\dots,t-2u\}$ and $Q=\{1,\dots,t-u\}$. Of course there are also the mirror relations of (2.7), namely the ones obtained switching columns by rows. We will denote them by $\mathbf{g}'_u$.

Remark 2.11. As already noticed, the highest bi-weight vector of $(\gamma_u^t|\lambda_u^t)$ is a trivial extension of the highest bi-weight vector of the same irreducible G-representation relative to $2u$-minors, namely $(\gamma_u^{2u}|\lambda_u^{2u})$. In this case $\mathbf{g}_u$ assumes the following simpler form:

$$\mathbf{g}_u = \sum_{\substack{A,B,C\\L,M,N}} (-1)^{A,B,C}(-1)^{L,M,N}[K\setminus A|1,\dots,u,L][K\setminus B|1,\dots,u,M][K\setminus C|1,\dots,u,N],$$

where the sum runs over the 3-partitions A,B,C of $K=\{1,\dots,u\}$ and L,M,N of $\{u+1,\dots,4u\}$ such that $|A|=|B|=|C|=|L|=|M|=|N|=u$ and A,B,C are ordered lexicographically.

2.4.2. *Highest bi-weight vectors of odd cubics.* Let u be a positive integer in $\{2,\dots,\lceil t/2\rceil\}$. We are going to describe the highest weight vector of one of the copies of

$$L_{\rho_u}V \subseteq \bigotimes^3\left(\bigwedge^t V\right).$$

To this aim, let us set

$$v_1 = \sum_{A,B,C} (-1)^{A,B,C}(e_P\wedge e_{K\setminus A})\otimes(e_P\wedge e_{K\setminus B}\wedge e_{t+u})\otimes(e_P\wedge e_{K\setminus C})$$

and

$$v_2 = \sum_{A,B,C} (-1)^{A,B,C}(e_P\wedge e_{K\setminus B}\wedge e_{t+u})\otimes(e_P\wedge e_{K\setminus A})\otimes(e_P\wedge e_{K\setminus C}),$$

where the sums run over the partitions A,B,C of $K=\{t-2u+2,\dots,t+u-1\}$ such that $|A|=|C|=u-1$ and $|B|=u$. Moreover, $P=\{1,\dots,t-2u+1\}$. It is not difficult to show that the element

$$v = v_1 - v_2 \in \bigotimes^3\left(\bigwedge^t V\right) \tag{2.8}$$

is a nonzero $\mathrm{U}_-(V)$-invariant. Moreover, since v has weight ${}^{\mathrm{t}}\rho_u$, it is the highest weight vector of one of the copies of $L_{\rho_u}V$.

In the same vein, let $u\in\{2,\dots,\lceil t/2\rceil\}$. Analogously to above, we set

$$w_1 = \sum_{L,M,N} (-1)^{L,M,N}(f_Q^*\wedge f_L^*\wedge f_{t-u+1}^*)\otimes(f_Q^*\wedge f_M^*)\otimes(f_Q^*\wedge f_N^*\wedge f_{t-u+1}^*)$$

and

$$w_2 = \sum_{L,M,N} (-1)^{L,M,N} (f^*_Q \wedge f^*_M) \otimes (f^*_Q \wedge f^*_L \wedge f^*_{t-u+1}) \otimes (f^*_Q \wedge f^*_N \wedge f^*_{t-u+1}),$$

where the sums run over the partitions L,M,N of $\{t-u+2,\dots,t+2u-1\}$ such that $|L| = |N| = u-1$ and $|M| = u$. Furthermore, $Q = \{1,\dots,t-u\}$. Once again, it is not difficult to show that the element

$$w = w_1 - w_2 \in \bigotimes^3 \left(\bigwedge^t W^*\right) \tag{2.9}$$

is a nonzero $\mathrm{U}_+(W)$-invariant. Moreover, since w has weight ${}^{\mathrm{t}}\sigma_u$, it is the highest weight vector of one of the copies of $L_{\sigma_u} W^*$.

Now, as for the even relations, we tensor the row part (2.8) and the column part (2.9) together and pass to the symmetric power $(S_t)_3$. After some manipulations, we get:

$$\begin{aligned}\mathbf{h}_u = {} & \\ \sum_{\substack{A,B,C\\L,M,N}} & (-1)^{A,B,C}(-1)^{L,M,N}([P,K\setminus A|Q,L,t-u+1][P,K\setminus B,t+u|Q,M][P,K\setminus C|Q,N,t-u+1] \\ & - [P,K\setminus A|Q,M][P,K\setminus B,t+u|Q,L,t-u+1][P,K\setminus C|Q,N,t-u+1])\end{aligned} \tag{2.10}$$

where the sum runs over the 3-partitions A,B,C of $K = \{t-2u+2,\dots,t+u-1\}$ and L,M,N of $\{t-u+2,\dots,t+2u-1\}$ such that $|A| = |C| = |L| = |N| = u-1$, $|B| = |M| = u$ and A is less than C lexicographically. Moreover $P = \{1,\dots,t-2u+1\}$ and $Q = \{1,\dots,t-u\}$. Of course there are also the mirror relations of (2.10), namely the ones obtained switching columns by rows. We will denote them by $\mathbf{h}'_u$.

We believe that the relations found so far generate J_t. Despite of the rather limited evidence for this belief we formulate it as a conjecture:

Conjecture 2.12. *For all t,m,n the polynomials $\mathbf{f}_{u,v}$ of degree 2 and $\mathbf{g}_u, \mathbf{g}'_u, \mathbf{h}_u, \mathbf{h}'_u$ of degree 3 (as far as they are defined in $S_t(m,n)$) generate $J_t(m,n)$ as a G-ideal. Equivalently,*

$$\begin{aligned} J_t/(S_t)_1 J_t \cong & \bigoplus_{\substack{u,v\in\{0,\dots,t\}\\ u+v \text{ even}\\ u\neq v}} L_{\tau_u} V \otimes L_{\tau_v} W^* \oplus \bigoplus_{\substack{u\le m-t\\ 2u\le n-t}} L_{\gamma_u} \otimes L_{\lambda_u} W^* \oplus \bigoplus_{\substack{u\le m-t\\ 2u\le n-t+1}} L_{\rho_u} \otimes L_{\sigma_u} W^* \\ & \oplus \bigoplus_{\substack{u\le n-t\\ 2u\le m-t}} L_{\lambda_u} \otimes L_{\gamma_u} W^* \oplus \bigoplus_{\substack{u\le n-t\\ 2u\le m-t+1}} L_{\sigma_u} \otimes L_{\rho_u} W^*. \end{aligned}$$

It is remarkable that all the minimal relations we have found, are not only shape relations, but even of single $\bigwedge^t$-type.

Remark 2.13. (a) How far $J_t(m,n)$ is from the ideal generated by the degree 2 relations can be easily analyzed in the case $t=2$, $m=3$, $n=4$. In this case the ideal Q generated by the Plücker relations is a complete intersection ideal of height 6 and $Q = J_2(3,4) \cap P$ where P is a prime ideal generated by Q and $(S_2)_{(3,3|3,3)}$. In fact, there is an automorphism of $S_2(3,4)$ carrying $J_2(3,4)$ into P so that $S_2(3,4)/P \cong A_2(3,4)$. Furthermore for $t=2$, $m=3$, $n=5$ the ideal of quadrics in $J_2(3,5)$ generate an ideal whose codimension is smaller than that of $J_2(3,5)$ itself.

(b) It was shown in [5] that the ideal I generated by the Plücker relations and the degree 3 relations in the irreducible representation of the bi-shapes $(\gamma_1|\lambda_1)$ and $(\lambda_1|\gamma_1)$ satisfy the following property: $J_t(m,n)_P = I_P$ for all prime ideals $P \supset J_t(m,n)$ for which $(A_t)_P$ is non-singular. (The singular locus of A_t was also determined in [5].) This supports Conjecture 2.12 to some extent.

(c) Using the methods of Section 3.2, we have computed the relations of the algebra of 2-minors of a symmetric $n \times n$ matrix with $n \le 5$ rows. Surprisingly the ideal is generated in degree 2.

(d) On the other hand, De Negri [9, Theorem 1.4] proved that there are no degree 2 relations between $2t$-pfaffians of an alternating $n \times n$ matrix for arbitrary t and n in characteristic 0.

2.5. **Determinantal relations.** It turns out that the relations $\mathbf{g}_1$ are of determinantal type. In the following we want to indicate how to construct more such determinantal highest bi-weight vectors in J_t. They are closely related to the structure of

$$\bigwedge^d E \otimes \bigwedge^d F^*.$$

As usual by now, we (have) set $E = \bigwedge^t V$, $F = \bigwedge^t W$, and $H = \mathrm{GL}(E) \times \mathrm{GL}(F)$. The H-bi-shape associated with the above H-module is $(d|d)$.

If we order the canonical bases of E and F in such a way that this linear order extends the componentwise partial order on t-uples of the canonical bases in V and W, respectively, then the unipotent subgroup of G that we used to define U-invariants embeds naturally into the unipotent subgroup of H defined by the order of the base elements. Therefore H-U-invariants are in particular G-U-invariants (in self explaining notation). The H-U-invariant of shape $(d|d)$ is simply the d-minor of the matrix whose entries represent the pairs of the first d base vectors in E and F, respectively. It remains to fill the rows and columns of this d-minor in such a way that one obtains an element in J_t.

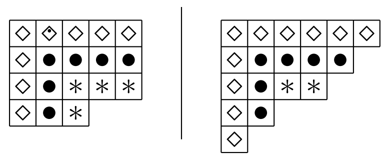
The crucial point is that the linear extension of the partial order is not unique (apart from trivial cases). Therefore we can choose different orders in E and F to produce asymmetric G-shapes in S_t, and these belong automatically to the ideal J_t of relations. In particular, the third largest element of a basis of E can be chosen in two ways, and this fact leads to the cubic relation $\mathbf{g}_1$.

We discuss the case $t = 2$ in detail. In each triangle of Example 2.14 below we take an initial subsequence of each row, and if no such subsequence sticks out further to the right than the one above it, the total sequence formed by concatenation represents an initial sequence in a suitable linear extension of the partial order. The entries of each subsequence represent a hook of type $(u+1,1,1,\dots,1) \vdash 2u$. The concatenated sequence represents a shape that is obtained by nesting these hooks, and thus we obtain $\mathrm{GL}(V)$-shapes in $\bigwedge^d E$.

Example 2.14. Let us consider the following two initial segments corresponding to two different linear extensions of the componentwise order:

$$
\begin{array}{ccccccc}
\diamond & \mathbf{12} & \mathbf{13} & \mathbf{14} & \mathbf{15} & 16 & \cdots \\
\bullet & & \mathbf{23} & \mathbf{24} & \mathbf{25} & 26 & \cdots \\
* & & & \mathbf{34} & \mathbf{35} & 36 & \cdots \\
 & & & & 45 & 46 & \cdots \\
 & & & & & & \cdots
\end{array}
\qquad
\begin{array}{ccccccc}
\diamond & \mathbf{12} & \mathbf{13} & \mathbf{14} & \mathbf{15} & \mathbf{16} & \cdots \\
\bullet & & \mathbf{23} & \mathbf{24} & \mathbf{25} & 26 & \cdots \\
* & & & \mathbf{34} & 35 & 36 & \cdots \\
 & & & & 45 & 46 & \cdots \\
 & & & & & & \cdots
\end{array}
$$

The elements of the initial segments are written in bold. The symbols at the beginning of the rows should help to understand how to get the following bi-shape from the two above initial segments:



The G-U-invariant of the above bi-shape is the determinant of the 9×9-matrix in Figure 3. Such a determinant is a degree 9 relation between 2-minors.

$$
\begin{pmatrix}
[12|12] & [12|13] & [12|14] & [12|15] & [12|16] & [12|23] & [12|24] & [12|25] & [12|34] \\
[13|12] & [13|13] & [13|14] & [13|15] & [13|16] & [13|23] & [13|24] & [13|25] & [13|34] \\
[14|12] & [14|13] & [14|14] & [14|15] & [14|16] & [14|23] & [14|24] & [14|25] & [14|34] \\
[15|12] & [15|13] & [15|14] & [15|15] & [15|16] & [15|23] & [15|24] & [15|25] & [15|34] \\
[23|12] & [23|13] & [23|14] & [23|15] & [23|16] & [23|23] & [23|24] & [23|25] & [23|34] \\
[24|12] & [24|13] & [24|14] & [24|15] & [24|16] & [24|23] & [24|24] & [24|25] & [24|34] \\
[25|12] & [25|13] & [25|14] & [25|15] & [25|16] & [25|23] & [25|24] & [25|25] & [25|34] \\
[34|12] & [34|13] & [34|14] & [34|15] & [34|16] & [34|23] & [34|24] & [34|25] & [34|34] \\
[35|12] & [35|13] & [35|14] & [35|15] & [35|16] & [35|23] & [35|24] & [35|25] & [35|34]
\end{pmatrix}
$$

FIGURE 3. A matrix representing a determinantal relation

Surprisingly, we have found the complete $\mathrm{GL}(V)$-decomposition of $\bigwedge^d E$ for $t = 2$: see [18, p. 65] for this classical plethysm.

3. UPPER BOUNDS ON THE DEGREE OF MINIMAL RELATIONS

In this section we will give some evidence for the truth of Conjecture 2.12. For $t = 2$ we have the strongest support: (i) the conjecture holds for $m \times n$-matrices with $m \leq 4$ and $m = n = 5$; (ii) the only minimal relations of degree 3 are those described in the conjecture; (iii) there are no minimal relations in degree 4. For $t = 3$ we have verified that there are no other minimal relations in degree 3. For arbitrary t, we can give some combinatorial support for the conjecture.

The results for $t = 2$ and $t = 3$ depend on computer calculations. For them an a priori bound on the degree of a minimal generator of J_t is very useful, and we will derive from the Castelnuovo-Mumford regularity of A_t.

3.1. **Castelnuovo-Mumford regularity of A_t.** For the computation of the Castelnuovo-Mumford regularity we will use the initial algebra $\mathrm{in}_\prec(A_t)$ of A_t with respect to a diagonal term order $\prec$ on R, i.e. a term order such that $\mathrm{in}_\prec([i_1 \dots i_p | j_1 \dots j_p]) = x_{i_1 j_1} \cdots x_{i_p j_p}$.

Theorem 3.1. *Apart from the cases discussed in Remark 1.2, we have:*

(i) *If $m+n-1 < \lfloor mn/t \rfloor$, then*

$$\mathrm{reg}(A_t) = mn - \lceil mn/t \rceil.$$

(ii) *if $m+n-1 \geq \lfloor mn/t \rfloor$, then*

$$\mathrm{reg}(A_t) = mn - \lfloor m(n+k_0)/t \rfloor.$$

where $k_0 = \lceil (tm+tn-mn)/(m-t) \rceil$.

Proof. We know that A_t is Cohen-Macaulay by [4, Theorem 7.10] and has dimension mn by [7, Proposition 10.16] because we have excluded the cases listed in Remark 1.2. Therefore we have $\mathrm{reg}(A_t) = \dim A_t + a(A_t) = mn + a(A_t)$. Here $a(A_t)$ is the a-invariant of A_t, i.e. the opposite of the least degree of a non-zero element of the graded canonical module of A_t. Since by [4, Theorem 7.10] $\mathrm{in}_\prec(A_t)$ is Cohen-Macaulay as well, we have $a(A_t) = a(\mathrm{in}_\prec(A_t))$. Hence it is enough to compute $a(\mathrm{in}_\prec(A_t))$. Denote by ω the canonical module of $\mathrm{in}_\prec(A_t)$. By [3, Lemma 3.3] ω is generated by the monomials of the form $\mathrm{in}_\prec(\Delta)$, where Δ is a product of minors of X of shape $\gamma = (\gamma_1, \dots, \gamma_h)$ where $|\gamma| = td$, $h < d$ and such that $\mathfrak{X} = \prod x_{ij}$ divides $\mathrm{in}_\prec(\Delta)$. Therefore, if d is the least number for which such a Δ exists, then $\mathrm{reg}(A_t) = mn - d$.

First let us consider case (i). Set $d_0 = \lceil mn/t \rceil$. Of course $\omega_d = 0$ if $d < d_0$. We have to show that $\omega_{d_0} \neq 0$. Let us pick the unique integer r_0 with $0 \leq r_0 < t$ and $mn + r_0 = d_0 t$. Of course we can consider a product $\Delta \in R$ of minors of shape $\gamma = (m^{n-m+1}, (m-1)^2, (m-2)^2, \dots, 1^2, r_0) \vdash d_0 t$ (possibly the partition has to be reordered but this does not matter) such that $\mathfrak{X}$ divides $\mathrm{in}_\prec(\Delta)$. (a) If $r_0 = 0$, then γ is a partition of $m+n-1$ rows: since $m+n-1 < d_0$ by hypothesis, we have $\mathrm{in}_\prec(\Delta) \in \omega$. (b) If $r_0 > 0$, the partition γ consists of $m+n$ rows. Then $d_0 = \lceil mn/t \rceil = \lfloor mn/t \rfloor + 1$, so the hypothesis implies $m+n < d_0$. Therefore also $\mathrm{in}_\prec(\Delta) \in \omega$ if $r_0 > 0$. We are done in case (i).

Now let us discuss case (ii). Notice that the integer k_0 introduced in (ii) is larger than 0. Let p_0 be the unique integer such that $0 \leq p_0 < t$ and $m(n+k_0) = d_0 t + p_0$. We can consider a product $\Delta \in R$ of minors of shape $\gamma = (m^{k_0+n-m}, (m-1)^2, \dots, 1^2, m-p_0)$ such that $\mathfrak{X}$ divides Δ. This is a partition of $d_0 t$ with $k_0+n+m-1$ parts. By the choice of k_0, one can verify that $k_0+n+m-1 < d_0$. So $\mathrm{in}_\prec(\Delta) \in \omega$, which implies $\omega_{d_0} \neq 0$.

To complete the proof showing that $\omega_d = 0$ whenever $d < d_0$, we need the following easy lemma.

Lemma 3.2. *With a little abuse of notation set $X = \{x_{ij} \ : \ i = 1, \dots, m, \ j = 1, \dots, n\}$. Define a poset structure on X in the following way:*

$$x_{ij} \leq x_{hk} \quad \text{if} \quad i = h \text{ and } j = k \quad \text{or} \quad i < h \text{ and } j < k.$$

Suppose that $X = X_1 \cup \dots \cup X_h$ where each X_i is a chain, i.e. any two elements of X_i are comparable, and set $N = \sum_{i=1}^h |X_i|$. Then

$$h \geq N/m + m - 1.$$

Let us take a product of minors $\Delta = \delta_1 \cdots \delta_h$ such that $\mathrm{in}_\prec(\Delta) \in \omega$. Let λ be the shape of Δ and suppose by contradiction that $|\lambda| = td$ with $d < d_0$. For $i = 1, \ldots, h$ set

$$X_i = \{x_{pr} : x_{pr} | \mathrm{in}_\prec(\delta_i)\}.$$

Since $\mathfrak{X}$ divides $\mathrm{in}_\prec(\Delta)$, with the notation of Lemma 3.2 we have that $X = \cup_{i=1}^h X_i$ where each X_i is a chain with respect to the order defined on X. So, by Lemma 3.2,

$$h \geq dt/m + m - 1.$$

We recall that $d_0 t = mn + mk_0 - p_0$, where $0 \leq p_0 < t$. Of course we can write $dt = mn + ms - q$ in a unique way, where $0 \leq q < m$. Before going on, notice that k_0 is the smallest natural number k satisfying the inequality

$$m + n + k - 1 < \left\lfloor \frac{m(n+k)}{t} \right\rfloor.$$

Of course $s \leq k_0$. There are two cases:

(i) If $s = k_0$, consider the inequalities

$$m + n + (s-1) - 1 = \frac{dt+q}{m} + m - 2 < \frac{dt}{m} + m - 1 \leq h \leq d - 1.$$

Notice that, since $d < d_0$, we have that $q \geq p_0 + t$. Moreover $m < 2t$, otherwise we would be in case (i) of the theorem. Thus

$$d - 1 = \frac{m(n+s) - q - t}{t} \leq \left\lfloor \frac{m(n+(s-1))}{t} \right\rfloor.$$

The inequalities above contradicts the minimality of k_0.

(ii) If $s < k_0$, then

$$n + s + m - 1 = \frac{dt+q}{m} + m - 1 \leq h < d = \frac{m(n+s) - q}{t} \leq \left\lfloor \frac{m(n+s)}{t} \right\rfloor.$$

Once again, this yields a contradiction to the minimality of k_0.

To sum up, we deduce that $\omega_d = 0$ whenever $d < d_0$, and this completes the proof. □

Remark 3.3. Let us look at the cases in Theorem 3.1.

(i) If X is a square matrix, that is $m = n$, one can easily check that we are in case (i) of Theorem 3.1 if and only if $m \geq 2t$.

(ii) The natural number k_0 of Theorem 3.1 may be very large. For instance, consider the case $t = m - 1$ and $n = m + 1$ with $m \geq 3$. One can easily check that we are in the case (ii) of Theorem 3.1. In this case we have $k_0 = m^2 - 2m - 1$. Therefore Theorem 3.1 yields

$$\mathrm{reg}(A_{m-1}(m, m+1)) = m.$$

Since $\mathrm{reg}(J_t) = \mathrm{reg}(A_t) + 1$ bounds the degree of a minimal generator of J_t from above, Theorem 3.1 yields an upper bound for the degree of a minimal relation between t-minors.

3.2. **Minimal relations between 2-minors of a $4 \times n$-matrix.** In this subsection we will indicate how to verify Conjecture 2.12 for $J_2(m,n)$ with $m \leq 4$ and $m = n = 5$. The following result enables us to succeed in this case by machine computation. It says that a minimal relation between t-minors of a $m \times n$-matrix must already "live" in a $m \times (m+t)$-matrix.

Theorem 3.4. *Let $(\gamma|\lambda)$ be a minimal representation in $J_t(m,n)$. Then $(\gamma|\lambda)$ is a minimal representation already in $J_t(m,m+t)$. In particular, if we denote the highest degree of a minimal generator of $J_t(m,n)$ by $d(t,m,n)$, then*

$$d(t,m,n) \leq d(t,m,m+t).$$

Proof. Suppose that $(\gamma|\lambda)$ is a minimal irreducible representation of $J_t(m,n)$. Then it is impossible that $(\gamma|\lambda)$ has only asymmetric bi-predecessors by Theorem 1.23. Since $\gamma_1 \leq m$, we must have $\lambda_1 \leq m+t$. Therefore it is a minimal irreducible representation in $J_t(m,m+t)$. ◻

The above theorem, together with Theorem 3.1, gives the following upper bound (far from what we have suggested in 2.12) for the degree of a minimal relation between t-minors.

Corollary 3.5. *The degree of a minimal generator of $J_t(m,n)$ is bounded above by*

$$m(m+t) - m - \left\lfloor \frac{m^2}{t} \right\rfloor + 1 \quad (\leq m^2 + (t-2)m).$$

However, Theorem 3.4 means that the validity of Conjecture 2.12 for 2-minors of a 3×5-matrix implies it for 2-minors of any $3 \times n$ matrix etc. In particular, Theorem 3.4 implies $d(2,3,n) \leq d(2,3,5)$ and $d(2,4,n) \leq d(2,4,6)$. Actually we can show that $d(2,3,5) \leq 3$ and $d(2,4,6) \leq 3$ by computer.

For Singular [10] the computation of $J_2(3,5)$ is a matter of seconds, but for $J_2(4,6)$ it is already a matter of days, and we succeeded only because of the following strategy that uses a priori informations on the Hilbert function of $A_2(m,n)$. Since the decomposition of the graded pieces of $A_t(m,n)$ can be computed easily via (1.2), an evaluation of the hook formula then yields its $\mathbb{k}$-dimension. (A tool for this computation had already been developed for [3].)

(1) Set $J = J_2(4,6)$, $S = S_2(4,6)$ and, for any $d \in \mathbb{N}$, let $J_{\leq d} \subseteq J$ denote the ideal generated by the polynomials in J of degree at most d. Corollary 3.5 implies that $J = J_{\leq 13}$.
(2) By elimination (for instance see Eisenbud [11, 15.10.4]), Singular computes a set of generators of $J_{\leq 3}$.
(3) For the degree reverse lexicographical term order, we compute a Gröbner basis of $J_{\leq 3}$ up to degree 13. So we get $B = \mathrm{in}_{\prec}(J_{\leq 3})_{\leq 13}$.
(4) The Hilbert function of S/B is easily computable, and we have

$$\mathrm{HF}_{S/J_{\leq 3}}(d) \leq \mathrm{HF}_{S/B}(d),$$

where equality holds for $d \leq 13$.

(5) Since $J_{\leq 3} \subseteq J$, we have $\mathrm{HF}_{S/B}(d) \geq \mathrm{HF}_{S/J}(d)$. However, comparing $\mathrm{HF}_{S/B}(d)$ with the precomputed $\mathrm{HF}_{S/J}(d)$ shows equality for $d \leq 13$. This implies $J_{\leq 3} = J_{\leq 13}$, and we are done.

The verification of $d(2,5,5) = 3$ is of similar complexity as that of $d(2,4,6) = 3$. However, already $d(2,5,6)$ or $d(3,4,7)$ seem to be out of reach for present day machines.

Theorem 3.6. *Conjecture 2.12 is true for* 2*-minors of a* $4 \times n$*-matrix and a* 5×5*-matrix. In particular, the only minimal relations between* 2*-minors of a* $4 \times n$*-matrix and a* 5×5*-matrix, respectively, are quadratics and cubics.*

The conjecture also holds for 3*-minors of a* 5×5*-matrix.*

Proof. Subsection 2.1 implies that the only degree 2 minimal generators of $J_t(m,n)$ are those listed in 2.12. The discussion above shows that there are no minimal generators of degree larger than 3 in $J_2(4,n)$, as predicted by Conjecture 2.12. It remains to show that the only degree 3 minimal generators are in the G-module generated by $\mathbf{g}_1$ and $\mathbf{g}_1'$. This will follow by a result of the next subsection, in which we prove this fact without restriction on m.

The statement on 3-minors of a 5×5-matrix follows from Proposition 1.3. □

3.3. **Cubic minimal relations between** 2**-minors.** In this subsection we are going to show that the only cubic minimal relations between 2-minors are those predicted in Conjecture 2.12, i.e. those in the G-space generated by $\mathbf{g}_1$ and by $\mathbf{g}_1'$. So we want to show that among the bi-diagrams $(\gamma|\lambda)$ in $\mathrm{Sym}^3\left(\bigwedge^2 V \otimes \bigwedge^2 W^*\right)$only $(\gamma_1|\lambda_1)$ (see (2.3) is minimal in $J_2(m,n)$. Since γ and λ are partitions of 6, the U-invariant of $(\gamma|\lambda)$ is in $S_2(6,6)$. This means that, for our task, it suffices to consider a 6×6-matrix. Since this format is presently unreachable by machine calculation, we must reduce it further.

Proposition 3.7. *Let* $t = 2$*. Then the following hold:*

(1) *The bi-shapes* $(2d|2d)$*,* $(2d-1,1|2d-1,1)$ *and* $(2d|2d-2,2)$ *have multiplicity* 1 *in* S_2 *(provided the vector space dimensions are sufficiently large).*
(2) *the bi-shape* $(2d|2d-1,1)$ *does not appear in* S_2*.*

Proof. In the following we use the plethysm (1.9). Let $E = \bigwedge^2 V$. Evidently $(2d)$ has multiplicity 1 in $\bigotimes^d E$, and since $(2d|2d)$ has multiplicity 1 in A_2, it must have multiplicity 1 in the intermediate S_2. Since $(2d)$ appears only in $\mathrm{Sym}^d E$ and $(2d-2,2)$ has multiplicity 1 in the latter, the multiplicity of $(2d|2d-2,2)$ in S_t must also be 1.

We claim that $(2d-1,1)$ is of single $\bigwedge^2$-type $\mu = (2,1,\dots,1) \vdash d$. In fact, $(2d-1,1)$ has multiplicity $d-1$ in $\bigotimes^d E$ by Pieri's rule, and this is also the multiplicity of μ in the $\mathrm{GL}(E)$-decomposition. Therefore it is enough that $(2d-1,1)$ appears in $L_\mu E$. Note that

$$\mathrm{Sym}^{d-1} E \otimes E = \mathrm{Sym}^d E \oplus L_\mu E,$$

the non-even successor $(2d-1,1)$ of $(2(d-1))$ must land in $L_\mu E$. Proposition 1.22 finishes the argument. □

Proposition 3.7 allows us to reduce the problem to size 4×5. The symmetric bi-shapes $(6|6)$ and $(5,1|5,1)$ have multiplicity 1 in S_2, occur in A_2 and so do not belong to J_2. The asymmetric shape $(6|5,1)$ is not represented in S_2 at all, and for the reduction to size

4×5 it remains to rule out the bi-shape $(6|4,2)$ of multiplicity 1, since the other bi-shapes involving (6) do not have symmetric bi-predecessors and $(5,1|5,1)$ has multiplicity 1.

We claim that $(S_t)_{(6|4,2)}$ is contained in the ideal generated by $(S_t)_{(4|2,2)}$. Because of Proposition 1.15 it is enough to prove this in $\mathrm{Sym}\big(\bigwedge^2 V\big) \sharp \mathrm{Sym}\big(\bigwedge^2 W^*\big)$. But in the Segre product it is enough to consider the single factors, and the algebra $\mathrm{Sym}\big(\bigwedge^2 V\big)$ is well-understood; see Abeasis and Del Fra [1].

For a 4×5-matrix it is not hard to check by machine computation that

$$\dim_{\Bbbk}(J_2)_3 = \dim_{\Bbbk}(((J_2)_{\le 2})_3) + \dim_{\Bbbk}\big((L_{\gamma_1}V\otimes L_{\lambda_1}W^*)\oplus(L_{\lambda_1}V\otimes L_{\gamma_1}W^*)\big)$$

where γ_1 and λ_1 are defined in (2.3). Thus the only subspace missing from $(((J_2)_{\le 2})_3$ is indeed the one predicted by Conjecture 2.12.

3.4. **No minimal degree 4 relations for 2-minors.** In this subsection we explain how to verify that there are no degree 4 minimal relations between 2-minors. The same method has been applied to exclude any further degree 3 minimal relations for 3 than those listed in Conjecture 2.12.

The first step is the computation of the $\mathrm{GL}(V)$-decomposition of $(S_2)_4$ by Lie. (The reader can reconstruct the decomposition from Table 2 and (1.8).) As documented above, it is already known that J_t is generated in degree 2 and 3 if $m=n=5$ or $m=4$. This excludes all bi-shapes from being minimal relations that fit into matrices of these sizes. After their exclusion and the exclusion of the cases covered by Theorem 3.4 and Proposition 3.7, there remain 6 critical bi-shapes of multiplicity 1 in J_2, and 2 other critical bi-shapes of multiplicity 2. (A further reduction would be possible via Theorem 1.23(ii).)

We want to show that they are not minimal relations. For multiplicity 1 it is enough to find a U-invariant of the given shape in $(S_2)_1\cdot(J_t)_3$. For example, let $(\gamma|\lambda)=(6,2|7,1)$. We try to “derive” it from $(\alpha|\beta)=(4,2|6)$. To this end we first compute $g_{\alpha\rightsquigarrow\gamma}$ and $g_{\beta\rightsquigarrow\lambda}$ by (1.7). Then we consider $g_{\alpha\rightsquigarrow\gamma}\otimes g_{\beta\rightsquigarrow\lambda}$ as an element of $\bigotimes^4(E\otimes F^*)$ (by reordering the factors) and pass to $\mathrm{Sym}^4(E\otimes F^*)$ by identifying summands that differ only by a simultaneous permutation of the E- and F^*-factors. The result, unless it is 0, is the desired U-invariant, and it could be found for all critical shapes of multiplicity 1. (Note that the computations depend on tableaux, not just diagrams, and not every choice of tableaux may work.)

If the critical shape has multiplicity 2, then we must derive two linearly independent U-invariants from asymmetric bi-shapes in degree 3. Again, this has turned out successful. The algorithm has been implemented by the authors in Singular. It is available with all input and output files from [6].

To justify the claim that $g_{\alpha\rightsquigarrow\gamma}\otimes g_{\beta\rightsquigarrow\lambda}$ indeed gives an element in $(S_2)_1J_t$, note that we take a sum of tensors $(\mathbb{Y}_A(a)\otimes a')\otimes(\mathbb{Y}_B(b)\otimes b')$ where A and B are tableaus of shapes α and β, respectively. Therefore $\mathbb{Y}_A(a)\otimes\mathbb{Y}_B(b)$ represents an element of J_2, and $a'\otimes b'$ represents an element of $(S_2)_1$.

A similar computation has been carried out for $t=3$ in order to exclude any further minimal degree 3 relations. It would certainly be possible to reach degree 6 for $t=2$ or degree 4 for $t=3$. However, then the algorithm must be re-implemented in a faster programming language, and its use must be further automatized.

As said in the introduction, we do not expect that relations are minimal because the algebra structure of S_t is too weak to exclude them from the ideal generated by the bi-predecessors that represent relations. If the following conjecture had a positive answer, then one would be a good deal closer to proving Conjecture 2.12. It reflects the computational experience described above.

Conjecture 3.8. *Let $(\gamma|\lambda)$ be a bi-shape occurring in $(S_t)_\mu$, and suppose that there exists a 1-predecessor μ' of μ that contains a t-bi-predecessor $(\alpha|\beta)$ of $(\gamma|\lambda)$. Then $(\gamma|\lambda)$ does occur in $(S_t)_1(S_t)_{(\alpha|\beta)}$.*

3.5. ***T*-shape relations.** In Theorem 2.4 we have identified cubic minimal relations in J_t that are even T-shape relations. In this subsection we want to show that these cubic relations and the degree 2 relations are the only T-shape relations in J_t. We recall that an asymmetric bi-shape $(\gamma|\lambda)$ is called a T-shape relation if it has only symmetric bi-predecessors of multiplicity 1 in T_t. This is a very strong condition:

Proposition 3.9. *Let γ,λ be (t,d)-admissible partitions. Then the following are equivalent:*

(i) *$(\gamma|\lambda)$ is a T-shape relation;*
(ii) *$(\gamma|\lambda)$ has a unique bi-predecessor;*
(iii) *γ and λ are both of multiplicity 1 in $\bigotimes^d \bigwedge V$ and, respectively, in $\bigotimes^d \bigwedge W^*$ and have the same predecessor.*

Proof. Let us just mention the main fact on which the easy proof relies. If γ or λ has more than one predecessor, then $(\gamma|\lambda)$ must have an asymmetric bi-predecessor in T_t, simply because we can pair any predecessors α and β of γ and λ, respectively, to a bi-predecessor $(\alpha|\beta)$ in T_t (but not necessarily in S_t!). This argument has already been used in the proof of Proposition 1.7. □

In view of Proposition 3.9 we must first classify the shapes of multiplicity 1 in $\bigotimes^d \bigwedge V$. To this end, we need the following lemma, whose proof is easy.

Lemma 3.10. *Let $\lambda = (\lambda_1,\dots,\lambda_k)$ be a diagram of $\bigotimes^d \bigwedge^t V$. Then λ has a unique predecessor if and only if either $\lambda_1 = \dots = \lambda_k$ (λ is a rectangle) or there exist i such that $\lambda_1 = \dots = \lambda_i > \lambda_{i+1} = \dots = \lambda_k$ and $k = d$ (λ is called a* fat hook*).*

Corollary 3.11. *For a diagram $\lambda = (\lambda_1,\dots,\lambda_k)$ of $\bigotimes^d \bigwedge^t V$, $d \geq 2$, the following are equivalent:*

(i) *λ has multiplicity 1 in $\bigotimes \bigwedge^t V$;*
(ii) *λ has a single predecessor λ', and λ' has again a single predecessor;*
(iii) *λ is a rectangle or fat hook of type (a) $\lambda_2 = \dots = \lambda_d$ or (b) $\lambda_1 = \dots = \lambda_{d-1}$.*

Remark 3.12. Diagrams λ of multiplicity 1 in $\bigotimes^d \bigwedge^t V$ are clearly of single $\bigwedge^t$-type μ where μ itself has multiplicity 1, and therefore represents either $\bigwedge^d(\bigwedge^t V)$ or $\mathrm{Sym}^d(\bigwedge^t V)$. We leave it to the reader to locate the diagrams in 3.11(iii).

The following theorem shows that we have found all T-shape relations. We suppress the case $d = 2$ since all asymmetric shapes of degree 2 are evidently T-shape relations.

Theorem 3.13. *The only shape relations of degree $d \geq 3$ are the cubics $(\gamma_u|\lambda_u)$ and $(\lambda_u|\gamma_u)$ where u varies in $\{1,\dots,\lfloor t/2\rfloor\}$.*

Proof. Let $(\gamma|\lambda)$ be a T-shape relation. We can assume that at least one of the two diagrams, say γ, is not a trivial extension, in other words has at most $d-1$ rows.

Suppose first that $\gamma_2 = \cdots = \gamma_d$. Since $\gamma_d = 0$, γ is a rectangle with one row of td boxes, and it is evident that we cannot find a second successor to the predecessor $(t(d-1))$ of γ that is different from γ but has itself multiplicity 1. (The only exception would be $d = 2$ in which case we could pair γ with $(2t-u,u)$.)

Now suppose that $\gamma_1 = \cdots = \gamma_{d-1}$. Since $\gamma_d = 0$ by assumption on γ, it must be a rectangle with $d-1 \geq 2$ rows. Again we look at the predecessor $\alpha = (\gamma_1,\dots,\gamma_{d-2},\gamma_{d-1}-t)$. Scanning the successors of α, we see that there is another successor $\lambda \neq \gamma$ of multiplicity 1 if and only if $d = 3$, t is even, and $\gamma_2 = 3t/2$. Then $\lambda = (2t,t/2,t/2)$, as desired. ☐

Remark 3.14. Let $(\gamma|\lambda)$ a bi-diagram in T_t and let $(\alpha_1|\beta_1),\dots,(\alpha_N|\beta_N)$ be its bi-predecessors counted with multiplicities in T_t (so it may happen that $(\alpha_i|\beta_i) = (\alpha_j|\beta_j)$ also if $i \neq j$). Suppose that exactly k of the bi-predecessors of $(\gamma|\lambda)$, say $(\alpha_1|\beta_1),\dots,(\alpha_k|\beta_k)$, are in K_t: If one of the copies of $L_\gamma V \otimes L_\lambda W^*$ is in K_t and does not belong to

$$\big((L_{\alpha_1}V \otimes L_{\beta_1}W^*) \oplus \cdots \oplus (L_{\alpha_k}V \otimes L_{\beta_k}W^*)\big) \otimes (T_t)_1 \\ \oplus (T_t)_1 \otimes \big((L_{\alpha_1}V \otimes L_{\beta_1}W^*) \oplus \cdots \oplus (L_{\alpha_k}V \otimes L_{\beta_k}W^*)\big),$$

then it is actually minimal in K_t. In particular, exploiting (1.2), a strategy to find minimal generators of K_t could be the following: to track down asymmetric bi-diagrams $(\gamma|\lambda)$ such that $k < N/2$ or symmetric ones such that $k < \lfloor N/2\rfloor$. However, one can easily realize that this situation happens if and only if $(\gamma|\lambda)$ is asymmetric, has multiplicity 1 in T_t and its unique bi-predecessor is symmetric. By Theorem 3.13, such a bi-diagram has to be among those predicted in Conjecture 2.12.

3.6. **No other degree 3 shape relations.** As usual let $E = \bigwedge^t V$ and $F = \bigwedge^t W$. In 2.2.2, we could found some minimal cubic relations between t-minors because the asymmetric bi-diagrams $(\rho_u|\sigma_u)$ in $\mathrm{Sym}^3(E \otimes F^*)$ have no asymmetric bi-predecessors in $\mathrm{Sym}^2(E \otimes F^*)$. Below we will show that, apart from $(\gamma_u|\lambda_u)$ and $(\rho_u|\sigma_u)$, no other bi-diagrams in $\mathrm{Sym}^3(E \otimes F^*)$ have this property. In other words, there exist no other degree 3 shape relations than the known ones. For the proof of this claim we need the following easy remark:

Remark 3.15. Suppose that λ is a $(t,3)$-admissible partition with k predecessors in $\bigotimes^2 E$, say a of them in $\mathrm{Sym}^2 E$ and the remaining $b = k-a$ in $\bigwedge^2 E$. Then $a-b \in \{-1,0,1\}$. To check this one has to use Lemma 2.1, noticing that

$$\tau_{u-1} \text{ and } \tau_{u+1} \text{ are predecessors of } \lambda \implies \tau_u \text{ is a predecessor of } \lambda.$$

Suppose that $(\gamma|\lambda)$ is an asymmetric bi-diagram in $\mathrm{Sym}^3(E \otimes F^*)$ such that γ has h predecessors and λ has k predecessors. We can assume that $1 \leq h \leq k$, because the issue is symmetric.

(i) Suppose that $h \geq 2$ and $k \geq 3$. Then, by Remark 3.15, at least one of $\mathrm{Sym}^2 E$ and $\bigwedge^2 E$ contains (at least) two predecessors of λ and one predecessor of γ. So in this case, we can deduce from (1.8) that $(\gamma|\lambda)$ has an asymmetric bi-predecessor which actually lives in $\mathrm{Sym}^2(E\otimes F^*)$.

(ii) Similar arguments finish the case $h=1$, $k\geq 4$.

(iii) If $h=k=1$, then we Theorem 3.13 implies: either $(\gamma|\lambda)=(\gamma_u|\lambda_u)$ for some u, or $(\gamma|\lambda)$ has an asymmetric bi-predecessor in $\bigotimes^2(E\otimes F^*)$. Moreover, since $(\gamma|\lambda)$ is in $\mathrm{Sym}^3(E\otimes F^*)$, such a bi-predecessor actually lives in $\mathrm{Sym}^2(E\otimes F^*)$.

We still need to deal with the cases $h=1$ and $k=2$, $h=1$ and $k=3$, $h=2$ and $k=2$. These cases are a bit more tricky:

Proposition 3.16. *Any asymmetric bi-diagram in* $\mathrm{Sym}^3(E\otimes F^*)$*, different from* $(\gamma_u|\lambda_u)$*,* $(\rho_u|\sigma_u)$ *and their mirror images, has an asymmetric bi-predecessor in* $\mathrm{Sym}^2(E\otimes F^*)$.

Proof. We keep the previous notation and continue with the remaining cases.

(i) $h=1$ and $k=2$. By Proposition 1.18, since $L_{(2,1)}F^*$ occurs with multiplicity 2 in $\bigotimes^3 F^*$, the irreducible $L_\lambda W^*$ occurs only in $L_{(2,1)}F^*$, and neither in Sym^3F^* nor in $\bigwedge^3 F^*$. On the other hand, since $h=1$, $L_\gamma V$ has to be in Sym^3E or in $\bigwedge^3 E$, but not in $L_{(2,1)}E$. Therefore $(\gamma|\lambda)$ cannot be in $\mathrm{Sym}^3(E\otimes F^*)$ by (1.8).

(ii) $h=1$ and $k=3$. Let us assume that 2 of the predecessors of λ are in Sym^2F^* and 1 in $\bigwedge^2 F^*$. The symmetric case is analogous, and there are no other cases by Remark 3.15. We claim that $L_\lambda W^*$ is not in $\bigwedge^3 F^*$. By Pieri's formula, we know that

$$\left(\bigwedge^2 F^*\right)\otimes F^*\cong\bigwedge^3 F^*\oplus L_{(2,1)}F^*.$$

Notice that one copy of $L_\lambda W^*$ is in $L_{(2,1)}F^*$ by Proposition 1.18. So, if $L_\lambda W^*$ were in $\bigwedge^3 F^*$, then λ would have 2 predecessors in $\bigwedge^2 F^*$, a contradiction.

It follows that $L_\lambda W^*$ does not occur in $\bigwedge^3 F^*$. Thus (1.8) implies that the only copy of $L_\gamma V$ has to be in Sym^3E, and the only predecessor of γ is in Sym^2E. Since λ has 2 predecessors in Sym^2E, $(\gamma|\lambda)$ has an asymmetric bi-predecessor which really lives in $\mathrm{Sym}^2(E\otimes F^*)$ by (1.8).

If $h=k=2$. We want to show that, in this case, there exist u and v such that $\gamma\in\{\rho_u,\sigma_u\}$ and $\lambda\in\{\rho_v,\sigma_v\}$. This is an immediate consequence of the following easy fact: A $(t,3)$-admissible diagram $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ has ℓ predecessors if and only if $\min\{\alpha_1-\alpha_2,\alpha_2-\alpha_3\}=\ell-1$. At this point, one can easily check that, apart from the cases in which $\gamma=\lambda$, $(\gamma|\lambda)=(\rho_u|\sigma_u)$ or $(\gamma|\lambda)=(\sigma_u|\rho_u)$, the bi-shape$(\gamma|\lambda)$ has always an asymmetric bi-predecessor or in $\mathrm{Sym}^2E\otimes\mathrm{Sym}^2F^*$, or in $\bigwedge^2 E\otimes\bigwedge^2 F^*$, and thus in $\mathrm{Sym}^2(E\otimes F^*)$ by (1.8). □

Remark 3.17. Using the plethysms computed by Lie we have checked that there are no other shape relations than the known degree 2 and 3 ones in the following cases: (i) $t=2,3$, $d\leq 5$ and (ii) $t=4,5$, $d\leq 4$.

References

[1] S. Abeasis and A. Del Fra, *Young diagrams and ideals of Pfaffians*, Adv. Math. 35 (1980), 158-178.
[2] W. Bruns, *Algebras defined by powers of determinantal ideals*, J. Algebra 142, 150-163, 1991).
[3] W. Bruns, A. Conca, *Algebras of minors*, J. Algebra 246, n. 1, pp. 311-330, 2001.

[4] W. Bruns, A. Conca, *Gröbner bases and determinantal ideals*, NATO Sci. Ser. II Math. Phys. Chem. 115, pp. 9-66, 2003.
[5] W. Bruns, A. Conca, *The variety of exterior powers of linear maps*, J. Algebra 322, pp. 2927-2949, 2009.
[6] W. Bruns, A. Conca and M. Varbaro, *Computations with Young symmetrizers*, available on line at http://www.http://www.home.uni-osnabrueck.de/wbruns/YoungSymm.
[7] W. Bruns, U. Vetter, *Determinantal rings*, Lecture notes in mathematics 1327, 1980.
[8] C. De Concini, D. Eisenbud, C. Procesi, *Young Diagrams and Determinantal Varieties*, Invent. math. 56, pp. 129-165, 1980.
[9] E. De Negri, *K-algebras generated by Pfaffians*, Math. J. Toyama Univ. 19 (1996), 105–114.
[10] W. Decker. G.M. Greuel, G. Pfister, H. Schönemann, *Singular - A computer algebra for polynomial computations*, available on line at http://www.singular.uni-kl.de.
[11] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer, 1994.
[12] W. Fulton, J. Harris, *Representation Theory. A First Course*, Graduate Texts in Mathematics 129, 1991.
[13] O. Holtz, B. Sturmfels, *Hyperdeterminantal relations among symmetric principal minors* , J. Algebra 316 (2007), 634–648.
[14] M.A.A. van Leeuwen, A.M. Cohen, B. Lissier, *LIE, A Package for Lie Group Computations*, Computer Algebra Nederland, ISBN 90-74116-02-7, 1992.
[15] S. Lin, B. Sturmfels, *Polynomial relations among principal minors of a* 4×4*-matrix*, J. Algebra 322 (2009), 4121–4131.
[16] L. Oeding, *Set-theoretic defining equations of the variety of principal minors of symmetric matrices*, Algebra and Number Theory 5 (2011), 75–109.
[17] C. Procesi, *Lie Groups. An Approach through Invariants and Representations*, University Lecture Series 8, 1996.
[18] J. M. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics 149, 2003.
[19] K. L. Whitehead, *Products of generalized determinantal ideals and decompositions under the action of general linear groups*, Ph.D. Thesis, University of Minnesota, 1982.

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