COMBINATORIAL SECANT VARIETIES
PART II

Bernd Sturmfels and Seth Sullivant

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We will see that the above facts are true for t-minors, too.

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Notice: $f \in I * J \Leftrightarrow f(\mathbf{y} + \mathbf{z}) \in I(\mathbf{y}) + J(\mathbf{z})$

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These and other properties about the join are proved in

A. Simis, B. Ulrich, *On the ideal of embedded join*, J. Alg. 226, 2000.

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where a subset of a poset is an antichain if it consists in incomparable elements.

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If $\mathrm{in}_\omega(I) = \mathrm{LT}_\prec(I)$ we say that ω represents $\prec$ for I.

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Since $\mathrm{in}_{(\omega,\omega,\omega)}(f(\mathbf{y}+\mathbf{z})) = m(\mathbf{y}+\mathbf{z})$, $f \in I * J \Rightarrow f(\mathbf{y}+\mathbf{z}) \in I(\mathbf{y}) + J(\mathbf{z})$

$\Rightarrow m(\mathbf{y}+\mathbf{z}) \in \mathrm{LT}_\prec(I)(\mathbf{y}) + \mathrm{LT}_\prec(J)(\mathbf{z})$

Initial ideal of joins

What is the relation between $\mathrm{LT}_\prec(I * J)$ and $LT_\prec(I) * \mathrm{LT}_\prec(J)$?

$$\mathrm{LT}_\prec(I_1 * I_2 * \cdots * I_r) \subseteq \mathrm{LT}_\prec(I_1) * \mathrm{LT}_\prec(I_2) * \cdots * \mathrm{LT}_\prec(I_r)$$

Pick $f \in I * J$, and set $m := \mathrm{LT}_\prec(f)$.

Take a vector $\omega \in \mathbb{Z}^n$ representing $\prec$ for I, J and f.

Consider the ideal $I(\mathbf{y}) + J(\mathbf{z}) \subseteq K[\mathbf{x}, \mathbf{y}, \mathbf{z}]$. We have

$$\begin{aligned} \mathrm{in}_{(\omega,\omega,\omega)}(I(\mathbf{y}) + J(\mathbf{z})) &= \mathrm{in}_{(\omega,\omega,\omega)}(I(\mathbf{y})) + \mathrm{in}_{(\omega,\omega,\omega)}(J(\mathbf{z})) \\ &= \mathrm{LT}_\prec(I)(\mathbf{y}) + \mathrm{LT}_\prec(J)(\mathbf{z}). \end{aligned}$$

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We say that $\prec$ is r-delightful for I if equality holds in (1). We say that $\prec$ is deligthful if it is r-delightful for any r.

EXAMPLES

Minors of a generic matrix

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$$I_r \subseteq I_2^{\{r-1\}}$$

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Moreover, as in the above case, one can show by hands that the 2-minors of X are a Gröbner basis of J_r with respect to a term order $\prec$ that favours the diagonals, and that there is a suitable poset P such that $\mathrm{LT}_{\prec}(J_2) = I(G(P))$.

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In short, copying the proof above we can show that $\prec$ is delightful for J_2 and that the r-minors of X form a Gröbner basis for J_r.

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For example, the 4-subfaffian associated to the minor $[i_1, i_2, i_3, i_4 \mid i_1, i_2, i_3, i_4]$ is the polynomial

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Notice that the above polynomials are the Plücker relations, and that $\mathrm{Pf}_4(X)$ is the defining ideal of the Grassmannian $\mathrm{Grass}(2, n)$.

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The rank of a skew-symmetric matrix is the maximum size of a non-vanishing subfaffian.

So, since the sum of $k-1$ skew-symmetric matrices of rank ≤ 2 is a skew-simmetric matrix of rank $\leq 2k-2$, we deduce that $\mathrm{Pf}_{2k}(X) \subseteq \mathrm{Pf}_4^{\{k-1\}}(X)$.

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It is not known wether higer Grassmannians, $\mathrm{Grass}(r, n)$ with $r \ge 3$, admit a delightful term order.

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The case of 2-minors of a generic matrix, which define the Segre embedding of two projective spaces, corresponds to one of the most simple Hibi rings, and we proved above that $\prec$ is delightful for it.

I think that it could be interesting to try to classify the distributive lattices for which $\prec$ is delightful, or at least to give a class for which it is.