Sequenze spettrali

A anello. Una **sequenza spettrale** $\mathbb{E}$ consiste nei seguenti dati:

- Una collezione di A-moduli $(E_r^{p,q})$, per $(p,q) \in \mathbb{Z}^2$ e $r \in \mathbb{Z}_{>0}$.
- Fissato r, per ogni (p,q), mappe di A-moduli

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

  tali che $d_r^{p,q} \circ d_r^{p-r,q+r-1} = 0$.
- Per ogni r, $E_{r+1}^{p,q} = \dfrac{\mathrm{Ker}(d_r^{p,q})}{\mathrm{Im}(d_r^{p-r,q+r-1})}$.

Conviene immaginarsi una sequenza spettrale come un libro, in cui a pagina r è illustrato il forglio $(E_r^{\bullet,\bullet})$ .....

Pagina 1 $\quad (d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q})$

$$\begin{array}{ccccccccccc}
\ddots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \iddots \\
\cdots & \longrightarrow & E_1^{0,3} & \longrightarrow & E_1^{1,3} & \longrightarrow & E_1^{2,3} & \longrightarrow & E_1^{3,3} & \longrightarrow & E_1^{4,3} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & E_1^{0,2} & \longrightarrow & E_1^{1,2} & \longrightarrow & E_1^{2,2} & \longrightarrow & E_1^{3,2} & \longrightarrow & E_1^{4,2} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & E_1^{0,1} & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} & \longrightarrow & E_1^{3,1} & \longrightarrow & E_1^{4,1} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0} & \longrightarrow & E_1^{3,0} & \longrightarrow & E_1^{4,0} & \longrightarrow & \cdots \\
\iddots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

Pagina 2 $\quad (d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1})$

$$
\begin{tikzcd}
\ddots \arrow[drr] & \vdots \arrow[drr] & \vdots \arrow[drr] & \vdots \arrow[drr] & \vdots \arrow[drr] & \vdots & \iddots \\
\cdots \arrow[drr] & E_2^{0,3} \arrow[drr] & E_2^{1,3} \arrow[drr] & E_2^{2,3} \arrow[drr] & E_2^{3,3} \arrow[drr] & E_2^{4,3} & \cdots \\
\cdots \arrow[drr] & E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} \arrow[drr] & E_2^{3,2} \arrow[drr] & E_2^{4,2} & \cdots \\
\cdots \arrow[drr] & E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & E_2^{3,1} \arrow[drr] & E_2^{4,1} & \cdots \\
\cdots \arrow[drr] & E_2^{0,0} \arrow[drr] & E_2^{1,0} \arrow[drr] & E_2^{2,0} \arrow[drr] & E_2^{3,0} \arrow[drr] & E_2^{4,0} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}
$$

Pagina 3 $\quad (d_3^{p,q} : E_3^{p,q} \to E_3^{p+3,q-2})$

$$
\begin{tikzcd}
\ddots \arrow[ddrrr] & \vdots \arrow[ddrrr] & \vdots \arrow[ddrrr] & \vdots \arrow[ddrrr] & \vdots & \vdots & \iddots \\
\cdots \arrow[ddrrr] & E_3^{0,3} \arrow[ddrrr] & E_3^{1,3} \arrow[ddrrr] & E_3^{2,3} \arrow[ddrrr] & E_3^{3,3} & E_3^{4,3} & \cdots \\
\cdots \arrow[ddrrr] & E_3^{0,2} \arrow[ddrrr] & E_3^{1,2} \arrow[ddrrr] & E_3^{2,2} \arrow[ddrrr] & E_3^{3,2} & E_3^{4,2} & \cdots \\
\cdots \arrow[ddrrr] & E_3^{0,1} \arrow[ddrrr] & E_3^{1,1} \arrow[ddrrr] & E_3^{2,1} \arrow[ddrrr] & E_3^{3,1} & E_3^{4,1} & \cdots \\
\cdots & E_3^{0,0} & E_3^{1,0} & E_3^{2,0} & E_3^{3,0} & E_3^{4,0} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}
$$

Diciamo che $\mathbb{E}$ è *eventualmente costante* se esiste r_0 tale che:

$$d_r^{p,q} = 0 \quad \forall \ r \geq r_0, \ (p,q) \in \mathbb{Z}^2.$$

Ciò è equivalente a dire che $E_r^{p,q} = E_{r_0}^{p,q} \ \forall \ r \geq r_0, \ (p,q) \in \mathbb{Z}^2$. In tal caso si dice che $\mathbb{E}$ *degenera* a pagina r_0, e si pone

$$E_\infty^{p,q} := E_{r_0}^{p,q} \ \forall \ (p,q) \in \mathbb{Z}^2.$$

OSS.: Per ogni $r \in \mathbb{Z}_{>0}$, consideriamo l'insieme

$$Q_r = \{(p,q) \in \mathbb{Z}^2 : E_r^{p,q} \neq 0\},$$

e si noti che $Q_1 \supset Q_2 \supset Q_3 \supset Q_4 \supset \ldots$. Dunque se Q_1 è un insieme limitato di $\mathbb{R}^2$, allora $\mathbb{E}$ è eventualmente costante.

DEF.: Diremo che $\mathbb{E}$ è *limitata* se Q_1 è un insieme limitato di $\mathbb{R}^2$.

(Quasi tutte le sequenze spettrali provenienti da situazioni "naturali" sono limitate).

Una sequenza spettrale $\mathbb{E}$ eventualmente costante *converge* ad una collezione di A-moduli $(E^n)_{n\in\mathbb{Z}}$ se: per ogni $n \in \mathbb{Z}$, esiste una filtrazione di A-sottomoduli di E^n:

$$\ldots \supset F^p E^n \supset F^{p+1} E^n \supset \ldots$$

tale che:

- $\bigcap_{p\in\mathbb{Z}} F^p E^n = \{0\}$ e $\bigcup_{p\in\mathbb{Z}} F^p E^n = E^n$;
- $E_\infty^{p,q} \cong F^p E^{p+q} / F^{p+1} E^{p+q} \ \forall \ (p,q) \in \mathbb{Z}^2$.

Solitamente, quello che succede è che per qualche motivo si conosce qualche pagina di $\mathbb{E}$ (solitamente la prima o la seconda). Quindi, se si sa che $(E_r^{p,q})_{(p,q)}$ è l'r-esima pagina di una sequenza spettrale convergente a $(E^n)_{n\in\mathbb{Z}}$, si scrive:

$$E_r^{p,q} \Rightarrow E^{p+q}.$$

OSS: Sia $\mathbb{E} = (E_r^{i,j})$ sequenza spettrale tale che $E_r^{p,q} \Rightarrow E^{p+q}$.

- Se A è un campo e $\mathbb{E}$ degenera a pagina r, allora:

$$E^n \cong \bigoplus_{p+q=n} E_r^{p,q}.$$

- $E_r^{p,q} = 0 \ \forall \ p+q=n \Rightarrow E^n = 0$.
- Se esiste p_0 tale che $E_r^{p,q} = 0 \ \forall \ p \neq p_0$, allora $\mathbb{E}$ degenera a pagina r e $E^n \cong E_r^{p_0, n-p_0}$.
- Se esiste q_0 tale che $E_r^{p,q} = 0 \ \forall \ q \neq q_0$ e $r \geq 2$, allora $\mathbb{E}$ degenera a pagina r e $E^n \cong E_r^{n-q_0, q_0}$.

Sia $C^\bullet$ un complesso di cocatene con differenziali $(d^n)_{n\in\mathbb{Z}}$. Ad ogni *filtrazione regolare* di $C^\bullet$ si può associare una sequenza spettrale convergente alla coomologia di $C^\bullet$:

DEF.: Una *filtrazione regolare* di $C^\bullet$ consiste in una filtrazione di A-sottomoduli di C^n (per ogni $n \in \mathbb{Z}$)

$$\ldots \supset F^p C^n \supset F^{p+1} C^n \supset \ldots$$

tale che:

- $\bigcap_{p\in\mathbb{Z}} F^p C^n = \{0\}$ e $\bigcup_{p\in\mathbb{Z}} F^p C^n = C^n$;
- $d^n(F^p C^n) \subset F^p C^{n+1}$.

TEOREMA: Ad ogni filtrazione regolare di $C^\bullet$ è associata una sequenza spettrale convergente ad $(E^n)_{n\in\mathbb{Z}}$ dove $E^n = H^n(C^\bullet)$. Inoltre $F^p E^n \cong H^n(F^p C^\bullet)$.

Dim.: Chi è interessato alla dimostrazione può leggerla alle pagine 203-206 del libro di Gelfand and Manin *Methods of homological algebra*.

Complessi doppi

Un **complesso doppio** di A-moduli $C^{\bullet,\bullet}$ consiste in:

- complessi di A-moduli $C^{\bullet,q}$ per ogni $q \in \mathbb{Z}$, con differenziali $(d_h^{n,q})_{n\in\mathbb{Z}}$;
- complessi di A-moduli $C^{p,\bullet}$ per ogni $p \in \mathbb{Z}$, con differenziali $(d_v^{p,n})_{n\in\mathbb{Z}}$,

tali che tutti i seguenti quadrati commutino:

$$\begin{tikzcd} C^{p,q+1} \arrow[r, "d_h^{p+1,q}"] & C^{p+1,q+1} \\ C^{p,q} \arrow[u, "d_v^{p,q}"] \arrow[r, "d_h^{p,q}"'] & C^{p+1,q} \arrow[u, "d_v^{p+1,q}"'] \end{tikzcd}$$

Il *complesso totale* di $C^{\bullet,\bullet}$ è il seguente complesso $\mathrm{Tot}(C)^\bullet$:

- $\mathrm{Tot}(C)^n = \bigoplus_{p+q=n} C^{p,q}$;
- $d^n : \mathrm{Tot}(C)^n \to \mathrm{Tot}(C)^{n+1}$ manda $x \in C^{p,n-p}$ in $d_h^{p,n-p}x + (-1)^p d_v^{p,n-p}x$.

$C^{\bullet,\bullet}$

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[u] \arrow[r, "d_h^{0,3}"] & C^{1,3} \arrow[u] \arrow[r, "d_h^{1,3}"] & C^{2,3} \arrow[u] \arrow[r, "d_h^{2,3}"] & C^{3,3} \arrow[u] \arrow[r, "d_h^{3,3}"] & C^{4,3} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[u, "d_v^{0,2}"'] \arrow[r, "d_h^{0,2}"] & C^{1,2} \arrow[u, "d_v^{1,2}"'] \arrow[r, "d_h^{1,2}"] & C^{2,2} \arrow[u, "d_v^{2,2}"'] \arrow[r, "d_h^{2,2}"] & C^{3,2} \arrow[u, "d_v^{3,2}"'] \arrow[r, "d_h^{3,2}"] & C^{4,2} \arrow[u, "d_v^{4,2}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[u, "d_v^{0,1}"'] \arrow[r, "d_h^{0,1}"] & C^{1,1} \arrow[u, "d_v^{1,1}"'] \arrow[r, "d_h^{1,1}"] & C^{2,1} \arrow[u, "d_v^{2,1}"'] \arrow[r, "d_h^{2,1}"] & C^{3,1} \arrow[u, "d_v^{3,1}"'] \arrow[r, "d_h^{3,1}"] & C^{4,1} \arrow[u, "d_v^{4,1}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[u, "d_v^{0,0}"'] \arrow[r, "d_h^{0,0}"] & C^{1,0} \arrow[u, "d_v^{1,0}"'] \arrow[r, "d_h^{1,0}"] & C^{2,0} \arrow[u, "d_v^{2,0}"'] \arrow[r, "d_h^{2,0}"] & C^{3,0} \arrow[u, "d_v^{3,0}"'] \arrow[r, "d_h^{3,0}"] & C^{4,0} \arrow[u, "d_v^{4,0}"'] \arrow[r] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}
$$

$\mathrm{Tot}(C)^\bullet$

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[u] \arrow[r, "d_h^{0,3}"] & C^{1,3} \arrow[u] \arrow[r, "d_h^{1,3}"] & C^{2,3} \arrow[u] \arrow[r, "d_h^{2,3}"] & C^{3,3} \arrow[u] \arrow[r, "d_h^{3,3}"] & C^{4,3} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[u, "d_v^{0,2}"'] \arrow[r, "d_h^{0,2}"] & C^{1,2} \arrow[u, "-d_v^{1,2}"'] \arrow[r, "d_h^{1,2}"] & C^{2,2} \arrow[u, "d_v^{2,2}"'] \arrow[r, "d_h^{2,2}"] & C^{3,2} \arrow[u, "-d_v^{3,2}"'] \arrow[r, "d_h^{3,2}"] & C^{4,2} \arrow[u, "d_v^{4,2}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[u, "d_v^{0,1}"'] \arrow[r, "d_h^{0,1}"] & C^{1,1} \arrow[u, "-d_v^{1,1}"'] \arrow[r, "d_h^{1,1}"] & C^{2,1} \arrow[u, "d_v^{2,1}"'] \arrow[r, "d_h^{2,1}"] & C^{3,1} \arrow[u, "-d_v^{3,1}"'] \arrow[r, "d_h^{3,1}"] & C^{4,1} \arrow[u, "d_v^{4,1}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[u, "d_v^{0,0}"'] \arrow[r, "d_h^{0,0}"] & C^{1,0} \arrow[u, "-d_v^{1,0}"'] \arrow[r, "d_h^{1,0}"] & C^{2,0} \arrow[u, "d_v^{2,0}"'] \arrow[r, "d_h^{2,0}"] & C^{3,0} \arrow[u, "-d_v^{3,0}"'] \arrow[r, "d_h^{3,0}"] & C^{4,0} \arrow[u, "d_v^{4,0}"'] \arrow[r] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}
$$

Dalla definizione abbiamo immediatamente *due* filtrazioni regolari di $\mathrm{Tot}(C)^\bullet$:

(i) ${}'F^p \,\mathrm{Tot}(C)^\bullet$ con ${}'F^p \,\mathrm{Tot}(C)^n = \bigoplus_{i \geq p} C^{i,n-i}$ per ogni $n \in \mathbb{Z}$;

(ii) ${}''F^q \,\mathrm{Tot}(C)^\bullet$ con ${}''F^q \,\mathrm{Tot}(C)^n = \bigoplus_{j \geq q} C^{n-j,j}$ per ogni $n \in \mathbb{Z}$.

Come visto, dunque queste filtrazioni danno luogo a due sequenze spettrali, rispettivamente $({}'E_r^{p,q})$ e $({}''E_r^{p,q})$, *entrambe convergenti alla coomologia di* $\mathrm{Tot}(C)^\bullet$.

Il nostro scopo ora è quello di descrivere le *seconde* pagine di tali sequenze spettrali in termini di $C^{\bullet,\bullet}$, ma prima diamo un'occhiata alle filtrazioni descritte sopra...

${}'F^0\,Tot(C)^\bullet$

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[u] \arrow[r, "d_h^{0,3}"] & C^{1,3} \arrow[u] \arrow[r, "d_h^{1,3}"] & C^{2,3} \arrow[u] \arrow[r, "d_h^{2,3}"] & C^{3,3} \arrow[u] \arrow[r, "d_h^{3,3}"] & C^{4,3} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[u, "d_v^{0,2}"'] \arrow[r, "d_h^{0,2}"] & C^{1,2} \arrow[u, "-d_v^{1,2}"'] \arrow[r, "d_h^{1,2}"] & C^{2,2} \arrow[u, "d_v^{2,2}"'] \arrow[r, "d_h^{2,2}"] & C^{3,2} \arrow[u, "-d_v^{3,2}"'] \arrow[r, "d_h^{3,2}"] & C^{4,2} \arrow[u, "d_v^{4,2}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[u, "d_v^{0,1}"'] \arrow[r, "d_h^{0,1}"] & C^{1,1} \arrow[u, "-d_v^{1,1}"'] \arrow[r, "d_h^{1,1}"] & C^{2,1} \arrow[u, "d_v^{2,1}"'] \arrow[r, "d_h^{2,1}"] & C^{3,1} \arrow[u, "-d_v^{3,1}"'] \arrow[r, "d_h^{3,1}"] & C^{4,1} \arrow[u, "d_v^{4,1}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[u, "d_v^{0,0}"'] \arrow[r, "d_h^{0,0}"] & C^{1,0} \arrow[u, "-d_v^{1,0}"'] \arrow[r, "d_h^{1,0}"] & C^{2,0} \arrow[u, "d_v^{2,0}"'] \arrow[r, "d_h^{2,0}"] & C^{3,0} \arrow[u, "-d_v^{3,0}"'] \arrow[r, "d_h^{3,0}"] & C^{4,0} \arrow[u, "d_v^{4,0}"'] \arrow[r] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}
$$

${}'F^1 \operatorname{Tot}(C)^\bullet$

$$\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[r] \arrow[u] & C^{1,3} \arrow[r, "d_h^{1,3}"] \arrow[u] & C^{2,3} \arrow[r, "d_h^{2,3}"] \arrow[u] & C^{3,3} \arrow[r, "d_h^{3,3}"] \arrow[u] & C^{4,3} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[r] \arrow[u] & C^{1,2} \arrow[r, "d_h^{1,2}"] \arrow[u, "-d_v^{1,2}"'] & C^{2,2} \arrow[r, "d_h^{2,2}"] \arrow[u, "d_v^{2,2}"'] & C^{3,2} \arrow[r, "d_h^{3,2}"] \arrow[u, "-d_v^{3,2}"'] & C^{4,2} \arrow[r] \arrow[u, "d_v^{4,2}"'] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[r] \arrow[u] & C^{1,1} \arrow[r, "d_h^{1,1}"] \arrow[u, "-d_v^{1,1}"'] & C^{2,1} \arrow[r, "d_h^{2,1}"] \arrow[u, "d_v^{2,1}"'] & C^{3,1} \arrow[r, "d_h^{3,1}"] \arrow[u, "-d_v^{3,1}"'] & C^{4,1} \arrow[r] \arrow[u, "d_v^{4,1}"'] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[r] \arrow[u] & C^{1,0} \arrow[r, "d_h^{1,0}"] \arrow[u, "-d_v^{1,0}"'] & C^{2,0} \arrow[r, "d_h^{2,0}"] \arrow[u, "d_v^{2,0}"'] & C^{3,0} \arrow[r, "d_h^{3,0}"] \arrow[u, "-d_v^{3,0}"'] & C^{4,0} \arrow[r] \arrow[u, "d_v^{4,0}"'] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}$$

${}'F^2\,\mathrm{Tot}(C)^\bullet$

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[r] \arrow[u] & C^{1,3} \arrow[r] \arrow[u] & C^{2,3} \arrow[r, "d_h^{2,3}"] \arrow[u] & C^{3,3} \arrow[r, "d_h^{3,3}"] \arrow[u] & C^{4,3} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[r] \arrow[u] & C^{1,2} \arrow[r] \arrow[u] & C^{2,2} \arrow[r, "d_h^{2,2}"] \arrow[u, "d_v^{2,2}"'] & C^{3,2} \arrow[r, "d_h^{3,2}"] \arrow[u, "-d_v^{3,2}"'] & C^{4,2} \arrow[r] \arrow[u, "d_v^{4,2}"'] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[r] \arrow[u] & C^{1,1} \arrow[r] \arrow[u] & C^{2,1} \arrow[r, "d_h^{2,1}"] \arrow[u, "d_v^{2,1}"'] & C^{3,1} \arrow[r, "d_h^{3,1}"] \arrow[u, "-d_v^{3,1}"'] & C^{4,1} \arrow[r] \arrow[u, "d_v^{4,1}"'] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[r] \arrow[u] & C^{1,0} \arrow[r] \arrow[u] & C^{2,0} \arrow[r, "d_h^{2,0}"] \arrow[u, "d_v^{2,0}"'] & C^{3,0} \arrow[r, "d_h^{3,0}"] \arrow[u, "-d_v^{3,0}"'] & C^{4,0} \arrow[r] \arrow[u, "d_v^{4,0}"'] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}
$$

$'F^3\,\mathrm{Tot}(C)^\bullet$

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[u] \arrow[r] & C^{1,3} \arrow[u] \arrow[r] & C^{2,3} \arrow[u] \arrow[r] & C^{3,3} \arrow[u] \arrow[r, "d_h^{3,3}"] & C^{4,3} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[u] \arrow[r] & C^{1,2} \arrow[u] \arrow[r] & C^{2,2} \arrow[u] \arrow[r] & C^{3,2} \arrow[u, "-d_v^{3,2}"'] \arrow[r, "d_h^{3,2}"] & C^{4,2} \arrow[u, "d_v^{4,2}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[u] \arrow[r] & C^{1,1} \arrow[u] \arrow[r] & C^{2,1} \arrow[u] \arrow[r] & C^{3,1} \arrow[u, "-d_v^{3,1}"'] \arrow[r, "d_h^{3,1}"] & C^{4,1} \arrow[u, "d_v^{4,1}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[u] \arrow[r] & C^{1,0} \arrow[u] \arrow[r] & C^{2,0} \arrow[u] \arrow[r] & C^{3,0} \arrow[u, "-d_v^{3,0}"'] \arrow[r, "d_h^{3,0}"] & C^{4,0} \arrow[u, "d_v^{4,0}"'] \arrow[r] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}
$$

${}''F^0\,Tot(C)^\bullet$

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[r, "d_h^{0,3}"] \arrow[u] & C^{1,3} \arrow[r, "d_h^{1,3}"] \arrow[u] & C^{2,3} \arrow[r, "d_h^{2,3}"] \arrow[u] & C^{3,3} \arrow[r, "d_h^{3,3}"] \arrow[u] & C^{4,3} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[r, "d_h^{0,2}"] \arrow[u, "d_v^{0,2}"'] & C^{1,2} \arrow[r, "d_h^{1,2}"] \arrow[u, "-d_v^{1,2}"'] & C^{2,2} \arrow[r, "d_h^{2,2}"] \arrow[u, "d_v^{2,2}"'] & C^{3,2} \arrow[r, "d_h^{3,2}"] \arrow[u, "-d_v^{3,2}"'] & C^{4,2} \arrow[r] \arrow[u, "d_v^{4,2}"'] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[r, "d_h^{0,1}"] \arrow[u, "d_v^{0,1}"'] & C^{1,1} \arrow[r, "d_h^{1,1}"] \arrow[u, "-d_v^{1,1}"'] & C^{2,1} \arrow[r, "d_h^{2,1}"] \arrow[u, "d_v^{2,1}"'] & C^{3,1} \arrow[r, "d_h^{3,1}"] \arrow[u, "-d_v^{3,1}"'] & C^{4,1} \arrow[r] \arrow[u, "d_v^{4,1}"'] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[r, "d_h^{0,0}"] \arrow[u, "d_v^{0,0}"'] & C^{1,0} \arrow[r, "d_h^{1,0}"] \arrow[u, "-d_v^{1,0}"'] & C^{2,0} \arrow[r, "d_h^{2,0}"] \arrow[u, "d_v^{2,0}"'] & C^{3,0} \arrow[r, "d_h^{3,0}"] \arrow[u, "-d_v^{3,0}"'] & C^{4,0} \arrow[r] \arrow[u, "d_v^{4,0}"'] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}
$$

$''F^1\,Tot(C)^{\bullet}$

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[u] \arrow[r, "d_h^{0,3}"] & C^{1,3} \arrow[u] \arrow[r, "d_h^{1,3}"] & C^{2,3} \arrow[u] \arrow[r, "d_h^{2,3}"] & C^{3,3} \arrow[u] \arrow[r, "d_h^{3,3}"] & C^{4,3} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[u, "d_v^{0,2}"'] \arrow[r, "d_h^{0,2}"] & C^{1,2} \arrow[u, "-d_v^{1,2}"'] \arrow[r, "d_h^{1,2}"] & C^{2,2} \arrow[u, "d_v^{2,2}"'] \arrow[r, "d_h^{2,2}"] & C^{3,2} \arrow[u, "-d_v^{3,2}"'] \arrow[r, "d_h^{3,2}"] & C^{4,2} \arrow[u, "d_v^{4,2}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[u, "d_v^{0,1}"'] \arrow[r, "d_h^{0,1}"] & C^{1,1} \arrow[u, "-d_v^{1,1}"'] \arrow[r, "d_h^{1,1}"] & C^{2,1} \arrow[u, "d_v^{2,1}"'] \arrow[r, "d_h^{2,1}"] & C^{3,1} \arrow[u, "-d_v^{3,1}"'] \arrow[r, "d_h^{3,1}"] & C^{4,1} \arrow[u, "d_v^{4,1}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[u] \arrow[r] & C^{1,0} \arrow[u] \arrow[r] & C^{2,0} \arrow[u] \arrow[r] & C^{3,0} \arrow[u] \arrow[r] & C^{4,0} \arrow[u] \arrow[r] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}
$$

$''F^2\,Tot(C)^\bullet$

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[u] \arrow[r, "d_h^{0,3}"] & C^{1,3} \arrow[u] \arrow[r, "d_h^{1,3}"] & C^{2,3} \arrow[u] \arrow[r, "d_h^{2,3}"] & C^{3,3} \arrow[u] \arrow[r, "d_h^{3,3}"] & C^{4,3} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[u, "d_v^{0,2}"'] \arrow[r, "d_h^{0,2}"] & C^{1,2} \arrow[u, "-d_v^{1,2}"'] \arrow[r, "d_h^{1,2}"] & C^{2,2} \arrow[u, "d_v^{2,2}"'] \arrow[r, "d_h^{2,2}"] & C^{3,2} \arrow[u, "-d_v^{3,2}"'] \arrow[r, "d_h^{3,2}"] & C^{4,2} \arrow[u, "d_v^{4,2}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[u] \arrow[r] & C^{1,1} \arrow[u] \arrow[r] & C^{2,1} \arrow[u] \arrow[r] & C^{3,1} \arrow[u] \arrow[r] & C^{4,1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[u] \arrow[r] & C^{1,0} \arrow[u] \arrow[r] & C^{2,0} \arrow[u] \arrow[r] & C^{3,0} \arrow[u] \arrow[r] & C^{4,0} \arrow[u] \arrow[r] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}
$$

In un complesso doppio $C^{\bullet,\bullet}$ possiamo vedere le mappe orizzontali come morfismi di complessi:

$$\ldots \to C^{p-1,\bullet} \xrightarrow{d_h^{p-1,\bullet}} C^{p,\bullet} \xrightarrow{d_h^{p,\bullet}} C^{p+1,\bullet} \to \ldots$$

e quelle verticali come morfismi di complessi:

$$\ldots \to C^{\bullet,q-1} \xrightarrow{d_v^{\bullet,q-1}} C^{\bullet,q} \xrightarrow{d_v^{\bullet,q}} C^{\bullet,q+1} \to \ldots$$

Dunque abbiamo mappe indotte sulle coomologie .....

$C^{\bullet,\bullet}$

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots \arrow[r] & C^{0,3} \arrow[u] \arrow[r, "d_h^{0,3}"] & C^{1,3} \arrow[u] \arrow[r, "d_h^{1,3}"] & C^{2,3} \arrow[u] \arrow[r, "d_h^{2,3}"] & C^{3,3} \arrow[u] \arrow[r, "d_h^{3,3}"] & \cdots \\
\cdots \arrow[r] & C^{0,2} \arrow[u, "d_v^{0,2}"'] \arrow[r, "d_h^{0,2}"] & C^{1,2} \arrow[u, "d_v^{1,2}"'] \arrow[r, "d_h^{1,2}"] & C^{2,2} \arrow[u, "d_v^{2,2}"'] \arrow[r, "d_h^{2,2}"] & C^{3,2} \arrow[u, "d_v^{3,2}"'] \arrow[r, "d_h^{3,2}"] & \cdots \\
\cdots \arrow[r] & C^{0,1} \arrow[u, "d_v^{0,1}"'] \arrow[r, "d_h^{0,1}"] & C^{1,1} \arrow[u, "d_v^{1,1}"'] \arrow[r, "d_h^{1,1}"] & C^{2,1} \arrow[u, "d_v^{2,1}"'] \arrow[r, "d_h^{2,1}"] & C^{3,1} \arrow[u, "d_v^{3,1}"'] \arrow[r, "d_h^{3,1}"] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[u, "d_v^{0,0}"'] \arrow[r, "d_h^{0,0}"] & C^{1,0} \arrow[u, "d_v^{1,0}"'] \arrow[r, "d_h^{1,0}"] & C^{2,0} \arrow[u, "d_v^{2,0}"'] \arrow[r, "d_h^{2,0}"] & C^{3,0} \arrow[u, "d_v^{3,0}"'] \arrow[r, "d_h^{3,0}"] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \ddots
\end{tikzcd}
$$

Coomologia verticale

$$
\begin{array}{cccccccccccc}
\ddots & & \vdots & & \vdots & & \vdots & & \vdots & & \iddots \\
\cdots & \longrightarrow & H_v^3(C^{0,\bullet}) & \xrightarrow{d_h^{0,3}} & H_v^3(C^{1,\bullet}) & \xrightarrow{d_h^{1,3}} & H_v^3(C^{2,\bullet}) & \xrightarrow{d_h^{2,3}} & H_v^3(C^{3,\bullet}) & \longrightarrow & \cdots \\
\cdots & \longrightarrow & H_v^2(C^{0,\bullet}) & \xrightarrow{d_h^{0,2}} & H_v^2(C^{1,\bullet}) & \xrightarrow{d_h^{1,2}} & H_v^2(C^{2,\bullet}) & \xrightarrow{d_h^{2,2}} & H_v^2(C^{3,\bullet}) & \longrightarrow & \cdots \\
\cdots & \longrightarrow & H_v^1(C^{0,\bullet}) & \xrightarrow{d_h^{0,1}} & H_v^1(C^{1,\bullet}) & \xrightarrow{d_h^{1,1}} & H_v^1(C^{2,\bullet}) & \xrightarrow{d_h^{2,1}} & H_v^1(C^{3,\bullet}) & \longrightarrow & \cdots \\
\cdots & \longrightarrow & H_v^0(C^{0,\bullet}) & \xrightarrow{d_h^{0,0}} & H_v^0(C^{1,\bullet}) & \xrightarrow{d_h^{1,0}} & H_v^0(C^{2,\bullet}) & \xrightarrow{d_h^{2,0}} & H_v^0(C^{3,\bullet}) & \longrightarrow & \cdots \\
\iddots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

Coomologia orizzontale della coomologia verticale

$$
\begin{array}{cccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & H_h^0 H_v^3(C^{\bullet,\bullet}) & H_h^1 H_v^3(C^{\bullet,\bullet}) & H_h^2 H_v^3(C^{\bullet,\bullet}) & H_h^3 H_v^3(C^{\bullet,\bullet}) & \cdots \\
\cdots & H_h^0 H_v^2(C^{\bullet,\bullet}) & H_h^1 H_v^2(C^{\bullet,\bullet}) & H_h^2 H_v^2(C^{\bullet,\bullet}) & H_h^3 H_v^2(C^{\bullet,\bullet}) & \cdots \\
\cdots & H_h^0 H_v^1(C^{\bullet,\bullet}) & H_h^1 H_v^1(C^{\bullet,\bullet}) & H_h^2 H_v^1(C^{\bullet,\bullet}) & H_h^3 H_v^1(C^{\bullet,\bullet}) & \cdots \\
\cdots & H_v^0 H_h^0(C^{\bullet,\bullet}) & H_h^1 H_v^0(C^{\bullet,\bullet}) & H_h^2 H_v^0(C^{\bullet,\bullet}) & H_h^3 H_v^0(C^{\bullet,\bullet}) & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Coomologia orizzontale

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & H_h^0(C^{\bullet,3}) \arrow[u] & H_h^1(C^{\bullet,3}) \arrow[u] & H_h^2(C^{\bullet,3}) \arrow[u] & H_h^3(C^{\bullet,3}) \arrow[u] & \cdots \\
\cdots & H_h^0(C^{\bullet,2}) \arrow[u, "d_v^{0,2}"'] & H_h^1(C^{\bullet,2}) \arrow[u, "d_v^{1,2}"'] & H_h^2(C^{\bullet,2}) \arrow[u, "d_v^{2,2}"'] & H_h^3(C^{\bullet,2}) \arrow[u, "d_v^{3,2}"'] & \cdots \\
\cdots & H_h^0(C^{\bullet,1}) \arrow[u, "d_v^{0,1}"'] & H_h^1(C^{\bullet,1}) \arrow[u, "d_v^{1,1}"'] & H_h^2(C^{\bullet,1}) \arrow[u, "d_v^{2,1}"'] & H_h^3(C^{\bullet,1}) \arrow[u, "d_v^{3,1}"'] & \cdots \\
\cdots & H_h^0(C^{\bullet,0}) \arrow[u, "d_v^{0,0}"'] & H_h^1(C^{\bullet,0}) \arrow[u, "d_v^{1,0}"'] & H_h^2(C^{\bullet,0}) \arrow[u, "d_v^{2,0}"'] & H_h^3(C^{\bullet,0}) \arrow[u, "d_v^{3,0}"'] & \cdots \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots
\end{tikzcd}
$$

Coomologia verticale della coomologia orizzontale

$$
\begin{array}{cccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & H^3_v H^0_h(C^{\bullet,\bullet}) & H^3_v H^1_h(C^{\bullet,\bullet}) & H^3_v H^2_h(C^{\bullet,\bullet}) & H^3_v H^3_h(C^{\bullet,\bullet}) & \cdots \\
\cdots & H^2_v H^0_h(C^{\bullet,\bullet}) & H^2_v H^1_h(C^{\bullet,\bullet}) & H^2_v H^2_h(C^{\bullet,\bullet}) & H^2_v H^3_h(C^{\bullet,\bullet}) & \cdots \\
\cdots & H^1_v H^0_h(C^{\bullet,\bullet}) & H^1_v H^1_h(C^{\bullet,\bullet}) & H^1_v H^2_h(C^{\bullet,\bullet}) & H^1_v H^3_h(C^{\bullet,\bullet}) & \cdots \\
\cdots & H^0_v H^0_h(C^{\bullet,\bullet}) & H^0_v H^1_h(C^{\bullet,\bullet}) & H^0_v H^2_h(C^{\bullet,\bullet}) & H^0_v H^3_h(C^{\bullet,\bullet}) & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

TEOREMA: Col le notazioni, introdotte si ha:

$$'E_2^{p,q} = H_h^p H_v^q(C^{\bullet,\bullet}) \quad \text{e} \quad ''E_2^{p,q} = H_v^p H_h^q(C^{\bullet,\bullet}).$$

In particolare, $H_h^p H_v^q(C^{\bullet,\bullet}) \Rightarrow H^{p+q}(\mathrm{Tot}(C)^\bullet) \Leftarrow H_v^p H_h^q(C^{\bullet,\bullet})$.

Dim.: Chi è interessato alla dimostrazione può leggerla a pagina 209 di Gelfand-Manin, Proposizione 10.

Siano M ed N due A-moduli, e $P_\bullet \to M \to 0$ e $Q_\bullet \to N \to 0$ rispettive risoluzioni proiettive. Per essere coerenti con la notazione coomologica finora adottata, poniamo $P^i := P_{-i}$ e $Q^i := Q_{-i}$ per ogni $i \in \mathbb{Z}$. In questo modo, ad esempio, la risoluzione proiettiva di M è:

$$\ldots \to P^{-2} \to P^{-1} \to P^0 \to M \to 0.$$

Si consideri il complesso doppio $T^{\bullet,\bullet}$ tale che, per ogni $(i,j) \in \mathbb{Z}^2$:

- $T^{i,j} = P^i \otimes_A Q^j$;
- $T^{\bullet,j} = P^\bullet \otimes_A Q^j$ e $T^{i,\bullet} = P^i \otimes_A Q^\bullet$.

$T^{\bullet,\bullet}$

$$
\begin{tikzcd}
 & 0 & 0 & 0 & 0 & \\
\cdots \arrow[r] & P^{-3} \otimes_A Q^{0} \arrow[u] \arrow[r] & P^{-2} \otimes_A Q^{0} \arrow[u] \arrow[r] & P^{-1} \otimes_A Q^{0} \arrow[u] \arrow[r] & P^{0} \otimes_A Q^{0} \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & P^{-3} \otimes_A Q^{-1} \arrow[u] \arrow[r] & P^{-2} \otimes_A Q^{-1} \arrow[u] \arrow[r] & P^{-1} \otimes_A Q^{-1} \arrow[u] \arrow[r] & P^{0} \otimes_A Q^{-1} \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & P^{-3} \otimes_A Q^{-2} \arrow[u] \arrow[r] & P^{-2} \otimes_A Q^{-2} \arrow[u] \arrow[r] & P^{-1} \otimes_A Q^{-2} \arrow[u] \arrow[r] & P^{0} \otimes_A Q^{-2} \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & P^{-3} \otimes_A Q^{-3} \arrow[u] \arrow[r] & P^{-2} \otimes_A Q^{-3} \arrow[u] \arrow[r] & P^{-1} \otimes_A Q^{-3} \arrow[u] \arrow[r] & P^{0} \otimes_A Q^{-3} \arrow[u] \arrow[r] & 0 \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

Coomologia verticale

$$
\begin{tikzcd}
 & 0 & 0 & 0 & 0 & \\
\cdots \arrow[r] & P^{-3} \otimes_A N \arrow[r] & P^{-2} \otimes_A N \arrow[r] & P^{-1} \otimes_A N \arrow[r] & P^{0} \otimes_A N \arrow[r] & 0 \\
\cdots & 0 & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 & 0 \\
\iddots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Coomologia orizzontale della coomologia verticale

$$
\begin{array}{ccccc}
 & & 0 & 0 & 0 \\
\cdots & (L_3(- \otimes_A N))(M) & (L_2(- \otimes_A N))(M) & (L_1(- \otimes_A N))(M) & (L_0(- \otimes_A N))(M) \\
\cdots & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 \\
\iddots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

$T^{\bullet,\bullet}$

$$\begin{tikzcd}
& 0 & 0 & 0 & 0 & \\
\cdots \arrow[r] & P^{-3} \otimes_A Q^{0} \arrow[u] \arrow[r] & P^{-2} \otimes_A Q^{0} \arrow[u] \arrow[r] & P^{-1} \otimes_A Q^{0} \arrow[u] \arrow[r] & P^{0} \otimes_A Q^{0} \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & P^{-3} \otimes_A Q^{-1} \arrow[u] \arrow[r] & P^{-2} \otimes_A Q^{-1} \arrow[u] \arrow[r] & P^{-1} \otimes_A Q^{-1} \arrow[u] \arrow[r] & P^{0} \otimes_A Q^{-1} \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & P^{-3} \otimes_A Q^{-2} \arrow[u] \arrow[r] & P^{-2} \otimes_A Q^{-2} \arrow[u] \arrow[r] & P^{-1} \otimes_A Q^{-2} \arrow[u] \arrow[r] & P^{0} \otimes_A Q^{-2} \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & P^{-3} \otimes_A Q^{-3} \arrow[u] \arrow[r] & P^{-2} \otimes_A Q^{-3} \arrow[u] \arrow[r] & P^{-1} \otimes_A Q^{-3} \arrow[u] \arrow[r] & P^{0} \otimes_A Q^{-3} \arrow[u] \arrow[r] & 0 \\
\iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Coomologia orizzontale

$$
\begin{tikzcd}
 & 0 & 0 & 0 & 0 & \\
\cdots & 0 & 0 & 0 & M \otimes_A Q^0 \arrow[u] & 0 \\
\cdots & 0 & 0 & 0 & M \otimes_A Q^{-1} \arrow[u] & 0 \\
\cdots & 0 & 0 & 0 & M \otimes_A Q^{-2} \arrow[u] & 0 \\
\cdots & 0 & 0 & 0 & M \otimes_A Q^{-3} \arrow[u] & 0 \\
\iddots & \vdots & \vdots & \vdots & \vdots \arrow[u] &
\end{tikzcd}
$$

Coomologia verticale della coomologia orizzontale

$$
\begin{array}{cccccc}
 & 0 & 0 & 0 & 0 & \\
\cdots & 0 & 0 & 0 & (L_0(M \otimes_A -))(N) & 0 \\
\cdots & 0 & 0 & 0 & (L_1(M \otimes_A -))(N) & 0 \\
\cdots & 0 & 0 & 0 & (L_2(M \otimes_A -))(N) & 0 \\
\cdots & 0 & 0 & 0 & (L_3(M \otimes_A -))(N) & 0 \\
\iddots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Essendo ogni modulo proiettivo piatto, per ogni $(i,j) \in \mathbb{Z}^2$ si ha:

$$H^k(T^{i,\bullet}) = \begin{cases} P^i \otimes_A N & \text{se } k = 0 \\ 0 & \text{altrimenti} \end{cases} \qquad \text{(coomologia verticale)}$$

$$H^k(T^{\bullet,j}) = \begin{cases} M \otimes_A Q^j & \text{se } k = 0 \\ 0 & \text{altrimenti} \end{cases} \qquad \text{(coomologia orizzontale)}$$

Quindi la coomologia orizzontale della coomologia verticale:

$$'E_2^{i,j} = H_h^i H_v^j(T^{\bullet,\bullet}) = \begin{cases} (L_{-i}(- \otimes_A N))(M) & \text{se } j = 0 \\ 0 & \text{altrimenti} \end{cases}$$

e la coomologia verticale della coomologia orizzontale sarà:

$$''E_2^{i,j} = H_v^i H_h^j(T^{\bullet,\bullet}) = \begin{cases} (L_{-i}(M \otimes_A -))(N) & \text{se } j = 0 \\ 0 & \text{altrimenti} \end{cases}$$

Allora ${}'E_2^{i,j} = {}'E_\infty^{i,j}$, e ${}''E_2^{i,j} = {}''E_\infty^{i,j}$. Ma poiché le due sequenze spettrali convergono alla stessa cosa, cioè a $H^{i+j}(\mathrm{Tot}(T)^\bullet)$, devono essere uguali:

$$(L_i(-\otimes_A N))(M) \cong (L_i(M \otimes_A -))(N) \cong H^{-i}(\mathrm{Tot}(T)^\bullet) =: \mathrm{Tor}_i^A(M,N).$$

ESERCIZIO: 1. Dimostrare che $\mathrm{Tor}_i^A(M,N) \cong \mathrm{Tor}_i^A(N,M)$.

2. Usare lo stesso metodo per dimostrare che

$$(R^i(\mathrm{Hom}_A(M,-)))(N) \cong (R^i(\mathrm{Hom}_A(-,N)))(M) =: \mathrm{Ext}_A^i(M,N).$$