SOME ASYMPTOTIC MEASURES FOR POWERS OF DETERMINANTAL IDEALS

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Notation

Fix positive integers $t \leq m \leq n$, and consider the $m \times n$-matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & \cdots & x_{mn} \end{pmatrix},$$

whose entries are algebraically independent over a field K. Let $K[X]$ be the polynomial ring generated by the entries of X, and $I_t \subseteq K[X]$ the prime ideal defining the locus of rank $< t$ matrices in $\mathbb{A}^{mn}$. The ideal I_t is known to be generated by the t-minors of X.

Furthermore, let us fix K-vector spaces W of dimension m and V of dimension n, and the group $G = \mathrm{GL}(W) \times \mathrm{GL}(V)$.

The action of G on $K[X]$

The group G acts on X by extending the rule:

$$(A, B) \cdot X = AXB^{-1} \quad \forall\, (A, B) \in G = \mathrm{GL}(W) \times \mathrm{GL}(V).$$

Under such an action, the following K-vector spaces are stable:

(i) All the graded pieces of $K[X]$.

(ii) The ideal I_t.

(iii) All the powers I_t^s for $s \in \mathbb{N}$.

(iv) All the symbolic powers $I_t^{(s)}$ for $s \in \mathbb{N}$.

With the above action there is an isomorphism of G-modules:

$$K[X] \cong \bigoplus_{d \geq 0} \mathrm{Sym}^d(W \otimes V^*)$$

If $\mathrm{char}(K) = 0$, each graded piece $\mathrm{Sym}^d(W \otimes V^*)$ decomposes in irreducible G-modules. An explicit decomposition is known since Cauchy: Each irreducible G-submodule of $\mathrm{Sym}^d(W \otimes V^*)$ occurs once in the decomposition, and is parametrized by a partition of d:

$$\lambda = (\lambda_1, \ldots, \lambda_k) \vdash d \qquad \text{s. t.} \qquad m \geq \lambda_1 \geq \ldots \geq \lambda_k \geq 1$$

This supplies a beautiful structure of $K[X]$ as a K-vector space, that we are going to describe soon.

To the irreducible G-submodule of $\mathrm{Sym}^d(W \otimes V^*)$ associated to a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash d$, corresponds in $K[X]$ the K-vector subspace of the degree d-polynomials generated by the products:

$$\Delta = \delta_1 \cdots \delta_k,$$

where δ_i is a λ_i-minor of X for all $i = 1, \dots, k$.

We will say that such a product of minors Δ has shape λ. Let us denote such a vector space M_λ.

Products of minors are not linearly independent over K, however the previous representation theoretic interpretation gives us a way to identify a subset of products of minors which is indeed a K-basis of $K[X]$.

Standard monomial theory

First of all, we need a notation for r-minors ($r \leq m$):

$$[i_1, \dots, i_r | j_1, \dots, j_r] = \det \begin{pmatrix} x_{i_1,j_1} & \dots & x_{i_1,j_r} \\ \vdots & & \vdots \\ x_{i_r,j_1} & \dots & x_{i_r,j_r} \end{pmatrix}$$

Giving the following partial order to the set of minors:

$$[i_1, \dots, i_r | j_1, \dots, j_r] \leq [u_1, \dots, u_s | v_1, \dots, v_s] \Leftrightarrow$$
$$r \geq s,\ i_q \leq u_q,\ j_q \leq v_q\ \forall\ q \in \{1, \dots, s\}$$

a standard monomial means a product of minors $\Delta = \delta_1 \cdots \delta_k$ such that $\delta_1 \leq \delta_2 \leq \dots \leq \delta_k$.

(Doubilet, Rota, Stein): The standard monomials form a K-basis of $K[X]$ (for any field K).

The first K-basis of the polynomial ring one would consider is that consisting of monomials. Such a basis has the wonderful property to be closed under multiplication, feature that makes monomial ideals so simple.

The set of standard monomials is not closed under multiplication, however the recipe to write a product of standard monomials as a linear combination of standard monomials is beautiful. Such a recipe, firstly described by Doubilet, Rota and Stein, is what makes of $K[X]$ and its quotients by determinantal ideals algebras with straightening law.

The multiplicative structure

Going back to the characteristic 0 case, the Cauchy decomposition provides a good structure of $K[X]$ that is suitable for the study of determinantal ideals as a K-vector space, but it doesn't say much on the multiplicative structure.

That is, if $\lambda \vdash d$ and $\mu \vdash e$, as M_λ and M_μ are both G-stable, also $M_\lambda \cdot M_\mu$ is, so we can decompose it as

$$M_\lambda \cdot M_\mu = \bigoplus_{\alpha \in A(\lambda,\mu)} M_\alpha,$$

where $A(\lambda, \mu)$ is a subset of partitions of $d + e$.

The question is: What is $A(\lambda, \mu)$???

A complete answer has been given by Whitehead, who showed that the set $A(\lambda, \mu)$ is given by the Littlewood-Richardson rule for the tensor product of the Schur modules associated to λ and μ.

A partial answer was previously given by De Concini, Eisenbud and Procesi, who gave a decomposition of

$$\underbrace{M_{(t)} \cdot M_{(t)} \cdot M_{(t)} \cdots M_{(t)}}_{s \text{ times}}.$$

Being $M_{(t)}$ the K-vector space generated by the t-minors, the result of De Concini, Eisenbud and Procesi allows us to understand the powers of determinantal ideals.

(De Concini, Eisenbud and Procesi): If $\mathrm{char}(K) = 0$, a product of minors of shape $(\lambda_1, \ldots, \lambda_k) \vdash st$ belongs to I_t^s if and only if $k \leq s$.

The above statement is false in positive characteristic. However, Bruns could prove the following by using the straightening law:

(Bruns): Without restriction on the characteristic, a product of minors of shape $(\lambda_1, \ldots, \lambda_k) \vdash st$ belongs to $\overline{I_t^s}$ if and only if $k \leq s$.

The F-pure threshold of determinantal ideals

If $I \subseteq S = K[x_1, \dots, x_N]$ is a homogeneous ideal and $\mathrm{char}(K) = p > 0$, then we define

$$\nu_I(q) = \max\{r \in \mathbb{N} : I^r \not\subset \mathfrak{m}^{[q]}\},$$

where $\mathfrak{m}$ is the maximal irrelevant ideal and $q = p^e$.

The F-pure threshold of I is then defined as:

$$\mathrm{fpt}(I) = \lim_{q \to \infty} \frac{\nu_I(q)}{q}$$

(Miller, Singh, V): The F-pure threshold of determinantal ideals is:

$$\mathrm{fpt}(I_t) = \min\left\{ \frac{(m-k)(n-k)}{t-k} : k = 0, \dots, t-1 \right\}$$

Sketch of the proof

First, let us prove that

$$\mathrm{fpt}(I_t) \ \leq \ \frac{(m-k)(n-k)}{t-k} \ \ \forall \ k = 0, \ldots, t-1$$

Take δ_k and δ_t minors of size k and t respectively. Bruns' result $\Rightarrow$

$$\delta_k^{t-k-1}\delta_t \ \in \ \overline{I_{k+1}^{t-k}},$$

and hence $\delta_k^{t-k-1} I_t \ \subseteq \ \overline{I_{k+1}^{t-k}}$. By the Briançon-Skoda theorem there exists an integer N such that

$$\left(\delta_k^{t-k-1} I_t\right)^{N+\ell} \ \in \ I_{k+1}^{(t-k)\ell}$$

for each integer $\ell \geq 1$.

Localizing at the prime ideal I_{k+1} of $K[X]$, one has

$$I_t^{N+\ell} \subseteq I_{k+1}^{(t-k)\ell} K[X]_{I_{k+1}} \qquad \text{for each } \ell \geq 1.$$

Since $K[X]_{I_{k+1}}$ is a regular local ring of dimension $(m-k)(n-k)$:

$$I_t^{N+\ell} \subseteq I_{k+1}^{[q]} K[X]_{I_{k+1}}$$

for positive integers ℓ and $q = p^e$ satisfying

$$(t-k)\ell > (q-1)(m-k)(n-k).$$

By the flatness of the Frobenius, then, we have:

$$I_t^{N+\ell} \subseteq I_{k+1}^{[q]}$$

for all integers q, ℓ satisfying the above inequality. This implies that

$$\nu_{I_t}(q) \leq N + 1 + \frac{(q-1)(m-k)(n-k)}{t-k}.$$

Dividing by q and passing to the limit, one obtains

$$\operatorname{fpt}(I_t) \leq \frac{(m-k)(n-k)}{t-k}.$$

To show that

$$\operatorname{fpt}(I_t) \;\geq\; \min\left\{ \frac{(m-k)(n-k)}{t-k} : k = 0, \ldots, t-1 \right\}$$

one must exhibit some elements in suitable powers of I which do not belong to some Frobenius powers of $\mathfrak{m}$.

Let's give the idea of how to produce such elements in the example $t = 2, m = 3, n = 4$, where we have to prove that $\operatorname{fpt}(I_2) \geq 6$. Consider the element:

$$\Delta = [3|1] \cdot [2,3|1,2] \cdot [1,2,3|1,2,3] \cdot [1,2,3|2,3,4] \cdot [1,2|3,4] \cdot [1|4].$$

Bruns $\Rightarrow \Delta \in \overline{I_2^6}$. Because we can choose a suitable term order such that $\operatorname{in}(\Delta)$ is squarefree, we get that:

$$\Delta^{q-1} \in \left(\overline{I_2^6}\right)^{q-1} \setminus \mathfrak{m}^{[q]}$$

We can get rid of integral closure by picking $0 \neq f \in K[X]$ s. t.:

$$f \cdot \Delta^\ell \in I_2^{6\ell} \ \ \forall \ \ell \geq 1,$$

and by taking $\ell = q - 1 - \deg(f)$, we have that $f \cdot \Delta^\ell \notin \mathfrak{m}^{[q]}$.

So we get:

$$\nu_I(q) \geq \frac{6(q - 1 - \deg(f))}{q}.$$

Dividing by q and passing to the limit as $q \to \infty$, we get $\mathrm{fpt}(I_2) \geq 6$. $\square$