LADDER DETERMINANTAL VARIETIES AND THEIR SYMBOLIC BLOWUPS

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ABSTRACT. In this article we show that the symbolic Rees algebra of a mixed (two-sided) ladder determinantal ideal is strongly F-regular. Furthermore, we prove that the symbolic associated graded algebra of a mixed ladder determinantal ideal is F-pure. Finally, we show that ideals of the poset of minors of a generic matrix give rise to F-pure algebras with straightening law.

1. INTRODUCTION

Let $X = (x_{i,j})$ be a $k \times \ell$ generic matrix. A subset $L \subseteq X$ is a *ladder* if whenever $x_{i,j}, x_{i',j'} \in L$ with $i \leqslant i'$ and $j \geqslant j'$, we have $x_{u,v} \in L$ for every $i \leqslant u \leqslant i'$ and $j' \leqslant v \leqslant j$ (see Figure 1). The *unmixed ladder determinantal ideal* $I_t(L)$ associated to L and $t \in \mathbb{N}$ is the ideal of $\Bbbk[L]$ generated by the t-minors of L. In this article we focus on the more general class of *mixed ladder determinantal ideals* $I_{\mathbf{t}}(L)$ which allows the sizes of the minors $\mathbf{t} = (t_1, \dots, t_v) \in \mathbb{Z}^v_{>0}$ to vary (see §2.3), and on *ladder determinantal varieties* which are the varieties they determine.

Unmixed ladder determinantal rings $\Bbbk[L]/I_t(L)$ were first introduced by Abhyankar in the study of singularities of Schubert varieties [2]. Narasimhan [37] showed that these rings are integral domains by explicitly describing Gröbner bases for the ideals $I_t(L)$. Later, Herzog and Trung [26], and Conca [10] further exploited Gröbner techniques to show that these rings are Cohen-Macaulay and normal, respectively. Conca and Herzog [11] recovered these results by showing more generally that unmixed ladder determinantal rings have rational singularities if $\Bbbk$ has characteristic zero, or equivalently, that they are F-rational if $\Bbbk$ has characteristic $p \gg 0$. In the same paper, Conca and Herzog point out that it was unknown at the time whether they are F-pure in characteristic $p > 0$.

A ladder L is *one-sided* if the three 'corners' $x_{1,1}, x_{1,\ell}, x_{k,\ell}$ belong to L. Gonciulea and Miller [19] introduced the class of mixed one-sided ladder determinantal rings and showed that they also have rational singularities and therefore they are Cohen-Macaulay and normal. Glassbrener and Smith proved that complete intersection one-sided unmixed ladder determinantal rings are strongly F-regular [18], and Conca and Herzog later showed that this holds without the complete intersection assumption [11] (see also [28, 35]). On the other hand, one-sided mixed ladder determinantal varieties are affine neighborhoods of Schubert varieties [16], thus the fact that they satisfy the above properties can also be

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deduced from the corresponding results for Schubert varieties [39] (see also [31, Theorem 2.4.3]). In fact, Schubert varieties are globally F-regular [33]. Two-sided mixed ladder determinantal rings were defined by Gorla [20], who showed that they are integral domains and Cohen-Macaulay. Regarding their F-singularities, Robichaux [40] mentioned that Escobar, Fink, Rajchgot, and Woo proved that all two-sided mixed ladder determinantal varieties are 321-avoiding Kazhdan-Lusztig varieties. In that case, they would be strongly F-regular because Kazhdan-Lusztig varieties are open subsets of Schubert varieties and, as such, they are strongly F-regular [33]. To the best of our knowledge the work of Escobar, Fink, Rajchgot, and Woo is still in preparation and not publicly available. As we could not find any argument or reference to support this statement, we decided not to consider this as a known fact until their work is finalized and made available. As a consequence of our results on symbolic blowups, we obtain that two-sided mixed ladder determinantal ideals define F-pure rings (see Corollary 4.3) hence settling a case which, taking into account our previous remarks, was still open, and answering the question of Conca and Herzog [11, p. 122].

In this article we focus on symbolic blowup algebras associated to the most general setup: two-sided mixed ladder determinantal ideals, as defined by Gorla [20].

Rees algebras associated to a filtration are very important objects in commutative algebra and algebraic geometry. Symbolic Rees algebras, i.e., Rees algebras associated to the filtration of symbolic powers of an ideal, are particularly interesting; for instance, see [22] for a survey on this topic. The fact that they might not be Noetherian rings is related to counterexamples to Hilbert's 14^{th} problem, and makes them typically harder to handle than ordinary Rees algebras (i.e., Rees algebras associated to ordinary powers). We also point out that the Noetherianity of symbolic Rees algebras of pure height one ideals has close connections with the minimal model program [4, 13, 24], and with the conjectural equivalence between weak and strong F-regularity; see [1] for recent progress in this direction. The Noetherianity of symbolic Rees algebras associated to ladder determinantal ideals was known in the unmixed case by work of Bruns and Conca [7, Theorem 4.1], who proved that they admit a finite SAGBI basis with respect to an antidiagonal term order. Using a combinatorial operation on ladders, which we call *chamfering*, we show that they are Noetherian also in the mixed case (see Theorem 3.5). We point out that our approach does not guarantee that the symbolic Rees algebra associated to a ladder determinantal ideal $I = I_{\mathbf{t}}(L)$ admits a finite SAGBI basis with respect to some monomial order $\prec$. However, when it does, as a byproduct of our methods we also conclude that $\operatorname{in}_\prec\left(I^{(n)}\right) = \operatorname{in}_\prec(I)^{(n)}$ for all $n \in \mathbb{N}$ and $p = \operatorname{char}(\Bbbk) \gg 0$ (see Corollary 4.5).

Our first main result concerns F-singularities of symbolic Rees algebras of ladder determinantal ideals: we show that they are all strongly F-regular.

Theorem A (Theorem 4.7)**.** *Let $\Bbbk$ be a perfect field of characteristic $p > 0$ and L be a ladder. For $\mathbf{t} = (t_1, \dots, t_v) \in \mathbb{Z}^v_{>0}$ let $I_{\mathbf{t}}(L) \subseteq \Bbbk[L]$ be the corresponding mixed ladder determinantal ideal. The symbolic Rees algebra $\mathscr{R}^s(I_{\mathbf{t}}(L)) = \bigoplus_{n \geqslant 0} I_{\mathbf{t}}(L)^{(n)}$ is strongly F-regular.*

Next, we turn our attention to symbolic associated graded rings $\mathrm{gr}^s(I_{\mathbf{t}}(L)) = \bigoplus_{n\geqslant 0} I_{\mathbf{t}}(L)^{(n)}/I_{\mathbf{t}}(L)^{(n+1)}$. Such rings are typically worse behaved than symbolic Rees algebras; however, their singularities have closer connections to those of the quotient $\Bbbk[L]/I_{\mathbf{t}}(L)$, since the latter is a direct summand of $\mathrm{gr}^s(I_{\mathbf{t}}(L))$. The aforementioned result about strong F-regularity of two-sided mixed ladder determinantal varieties points to the direction that $\mathrm{gr}^s(I_{\mathbf{t}}(L))$ might always be strongly F-regular. While we could not establish whether this is true or not, we prove that $\mathrm{gr}^s(I_{\mathbf{t}}(L))$ is always at least F-pure.

Theorem B (Theorem 4.2)**.** *Let $\Bbbk$ be a perfect field of characteristic $p > 0$ and L be a ladder. For $\mathbf{t} = (t_1,\dots,t_v) \in \mathbb{Z}^v_{>0}$ let $I_{\mathbf{t}}(L) \subseteq \Bbbk[L]$ be the corresponding mixed ladder determinantal ideal. The symbolic associated graded ring $\mathrm{gr}^s(I_{\mathbf{t}}(L))$ is F-pure.*

At the end of Section 4 we show that two-sided unmixed ladder determinantal ideals are *Knutson* associated to the product f of the minors corresponding to the main antidiagonals of the ladder. The notion of Knutson ideals was introduced over finite fields [30] and later developed over any field (also of characteristic 0) by Seccia [41]. More generally we prove that the ideal generated by fixed-size minors in the subladders corresponding to consecutive rows or columns in the matrix are Knutson ideals associated to the polynomial f described above. In particular, the natural minimal generators of any sum of these ideals form a Gröbner basis with respect to an antidiagonal term order. If, besides the operation of taking sums, we also allow the one of taking intersections, all the resulting ideals still admit a squarefree initial ideal with respect to an antidiagonal term order and, in positive characteristic, define F-pure rings (Theorem 4.8).

When $L = X$ the above result had already been proved by Seccia [42]. In the case $L = X$ we also prove that any ideal of fixed-size minors of a top-left or bottom-right corner submatrix of X is Knutson associated to f (Theorem 5.3). As a consequence any sum of these ideals is Knutson associated to f, their natural minimal generators form a Gröbner basis with respect to antidiagonal term orders. If, besides the operation of taking sums, we also allow the one of taking intersections, all the resulting ideals still admit a squarefree initial ideal with respect to an antidiagonal term order and, in positive characteristic, define F-pure rings. Some classes of ideals that can be obtained this way and for which we obtain the latter consequences are Schubert determinantal ideals and, more generally, alternating sign matrix ideals as they are sums of ideals of minors of top-left corner submatrices of X (see Corollary 5.4). We remark that the fact that Schubert determinantal varieties are F-pure in positive characteristic has been known since the work of Brion and Kumar [5], and they are in fact strongly F-regular as already pointed out before. The fact that the natural minimal generators of Schubert determinantal ideals form a Gröbner basis with respect to antidiagonal term orders was also known by work of Knutson and Miller [31]. Moreover, a proof by induction on the Bruhat order, and so different from the one given here, of the fact that Schubert determinantal ideals are Knutson associated to f was already sketched by Knutson [30]. Klein and Weigandt [29] showed that alternating sign matrix ideals have radical initial ideals with respect to antidiagonal term orders and hence they are radical themselves. All of the above facts are recovered, and sometimes strengthened, by Corollary 5.4.

We conclude by highlighting a connection of our work with Eisenbud's suspicion that graded Algebras with Straightening Law (ASL for short) might always be F-pure in positive characteristic [15, p. 245]. As noticed by Koley and Varbaro [32, Remark 5.2] this is not always true, although all graded ASL are F-injective. We confirm Eisenbud's intuition for ASL arising from ideals of the poset of minors of a generic matrix: since they are intersection of sums of ideals of minors of top-left corner submatrices of X, our results yield that they are Knutson.

Theorem C (Corollary 5.6)**.** *Let $\Bbbk$ be a field, X be a generic matrix and Π be the poset of all minors of X. There exists a polynomial $f \in \Bbbk[X]$ such that, for any ideal $\Omega \subseteq \Pi$, the poset ideal $\Omega\Bbbk[X]$ is Knutson with respect to f. In particular, $\operatorname{in}_\prec(\Omega\Bbbk[X])$ is a squarefree monomial ideal for any antidiagonal term order $\prec$, and $\Bbbk[X]/\Omega\Bbbk[X]$ is F-pure in positive characteristic.*

An analogous result actually holds for what we call generalized poset ideals, a class which strictly includes poset ideals; see Corollary 5.10.

2. Background

In this section we include some preliminary material that is needed in the subsequent sections. In particular, in §2.3 and §2.4 we recall the definitions of the classes of determinantal ideals that are treated in this paper.

2.1. **Methods in prime characteristic.**

We begin by recalling some concepts from positive characteristic commutative algebra. For more information we refer the reader to Ma and Polstra's recent survey [34].

Definition 2.1. Let R be a ring of prime characteristic $p > 0$. For every $e \in \mathbb{Z}_{>0}$, we denote by $F^e_*(R) = \{F^e_*(r) \mid r \in R\}$ the R-algebra which is isomorphic to R as a ring and whose action by R is given by $fF^e_*(r) = F^e_*\left(f^{p^e} r\right) = F^e_*\left(f^{p^e}\right) F^e_*(r)$. We say that R is *F-finite* if the the Frobenius map $F^e : R \to F^e_*(R)$, $r \mapsto F^e_*(r^{p^e})$ is finite for some, or equivalently all, $e \in \mathbb{Z}_{>0}$, that is, if $F^e_*(R)$ is a finitely generated R-module.

We say that R is *F-pure* if for some, or equivalently all, $e \in \mathbb{Z}_{>0}$ the Frobenius map F^e is pure, i.e., the map $M \otimes_R R \xrightarrow{\mathbf{1}_M \otimes_R F^e} M \otimes_R F^e_*(R)$ is injective for every R-module M. We say that R is *F-split* if for some, or equivalently all, $e \in \mathbb{Z}_{>0}$ the map F^e splits. Clearly, if R is F-split, it is F-pure; the converse holds if R is F-finite. We also note that if R is F-pure, then it is reduced, and in this case one can identify the map $R \to F^e_*(R)$ with the natural inclusion $R \hookrightarrow R^{1/p^e}$, where R^{1/p^e} is the ring of p^e-th roots of R. In particular, R is F-pure (resp. F-split) if and only if the map $R \to R^{1/p^e}$ is pure (resp. splits) for some, or equivalently all, $e \in \mathbb{Z}_{>0}$.

We now recall the following notion that connects symbolic powers and the property of being F-split [14].

Definition 2.2. Let $R = \Bbbk[x_1, \ldots, x_d]$ be a standard graded polynomial ring over an F-finite field $\Bbbk$ of prime characteristic. We say that a homogeneous ideal $I \subseteq R$ is *symbolic F-split* if there exists a splitting $\phi : R^{1/p} \to R$ such that $\phi\left(\left(I^{(np+1)}\right)^{1/p}\right) \subseteq I^{(n+1)}$ for every $n \in \mathbb{N}$. Equivalently, if

$$\bigcap_{n=0}^{\infty} \left[\left(I^{(n+1)}\right)^{[p]} : I^{(np+1)}\right] \not\subseteq \mathfrak{m}^{[p]}.$$

Remark 2.3. In the present paper, ideals which are symbolic F-split are especially relevant because of the following two properties:

(1) If $I \subseteq R$ is symbolic F-split, then the *symbolic Rees algebra* $\mathscr{R}^s(I) := \oplus_{n\in\mathbb{N}} I^{(n)}$ and the *symbolic associated graded algebra* $\operatorname{gr}^s(I) := \oplus_{n\in\mathbb{N}} I^{(n)}/I^{(n+1)}$ are F-split. In particular, if I is symbolic F-split then R/I is also F-split [14, Theorem 4.7].
(2) Let $I \subseteq R$ be a homogeneous ideal and $H := \operatorname{bigheight}(I)$ the maximum height of a minimal prime ideal of I. If there exists a term order $\prec$ on R for which $\operatorname{in}_\prec\left(I^{(H)}\right)$ contains a squarefree monomial, then I is symbolic F-split [14, Lemma 6.2(1)]. Moreover, in this case $\operatorname{in}_\prec(I)$ is automatically radical [32, Theorem 3.13], and if I is equidimensional then both $\bigoplus_{n\in\mathbb{N}} \operatorname{in}_\prec(I^{(n)})$ and $\bigoplus_{n\in\mathbb{N}} \frac{\operatorname{in}_\prec(I^{(n)})}{\operatorname{in}_\prec\left(I^{(n+1)}\right)}$ are F-split [14, Proposition 7.5].

Let $I \subseteq R$ be a homogeneous ideal such that $\operatorname{in}_\prec(I)$ is radical. In this case, we have that $\operatorname{in}_\prec\left(I^{(n)}\right) \subseteq \operatorname{in}_\prec(I)^{(n)}$ for every $n \in \mathbb{N}$ [43, Proposition 5.1], and hence

$$\operatorname{in}_\prec(I)^n \subseteq \operatorname{in}_\prec\left(I^{(n)}\right) \subseteq \operatorname{in}_\prec(I)^{(n)} \quad \text{for every } n \in \mathbb{N}.$$

Since $\operatorname{in}_\prec(I)$ is radical, $\operatorname{in}_\prec\left(I^{(n)}\right) = \operatorname{in}_\prec(I)^{(n)}$ if and only if $\operatorname{in}_\prec\left(I^{(n)}\right)$ has no embedded components. This fact allows us to show the following interesting relationship between symbolic powers and initial ideals.

Proposition 2.4. *Let $R = \Bbbk[x_1, \ldots, x_d]$ be a standard graded polynomial ring over a perfect field $\Bbbk$ of prime characteristic. Let $P \subseteq R$ be a homogeneous prime ideal of height h. Fix a term order $\prec$ on R and assume that $\bigoplus_{n\in\mathbb{N}} \operatorname{in}_\prec\left(P^{(n)}\right)$ is Noetherian. If $\operatorname{char}(\Bbbk) \gg 0$, the following are equivalent*

(1) $\operatorname{in}_\prec\left(P^{(n)}\right) = \operatorname{in}_\prec(P)^{(n)}$ *for every $n \in \mathbb{N}$ and $\operatorname{in}_\prec(P)$ is radical.*
(2) $\operatorname{in}_\prec\left(P^{(h)}\right) = \operatorname{in}_\prec(P)^{(h)}$ *and $\operatorname{in}_\prec(P)$ is radical.*
(3) $\operatorname{in}_\prec\left(P^{(h)}\right)$ *contains a squarefree monomial.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. For (2) $\Rightarrow$ (3) we note that $x_1 \cdots x_d$ belongs to Q^h for every minimal prime ideal Q of $\operatorname{in}_\prec(P)$ and then $x_1 \cdots x_d \in \operatorname{in}_\prec(P)^{(h)} = \operatorname{in}_\prec\left(P^{(h)}\right)$. It remains to show

$(3) \Rightarrow (1)$. From Remark 2.3(2) it follows that $\mathrm{in}_\prec(P)$ is radical and $\bigoplus_{n\in\mathbb{N}} \frac{\mathrm{in}_\prec(P^{(n)})}{\mathrm{in}_\prec(P^{(n+1)})}$ is reduced. Thus, $\mathrm{in}_\prec\left(P^{(n)}\right) = \mathrm{in}_\prec(P)^{(n)}$ for every $n \in \mathbb{N}$ [14, Corollary 7.10]. $\square$

Remark 2.5. The lower bound on the characteristic of the field $\Bbbk$ in Proposition 2.4 can be made precise: if $\bigoplus_{n\in\mathbb{N}} \mathrm{in}_\prec(P^{(n)})$ is generated as an R-algebra by forms of degrees $b_1,\dots,b_m$, then Proposition 2.4 applies if $\mathrm{char}(\Bbbk) > \mathrm{lcm}(b_1,\dots,b_m)$.

2.2. **Knutson ideals.** The notion of *Knutson ideals* was coined by Conca and Varbaro [12] after Knutson's work on compatibly split ideals and degenerations [30] (see also [6]).

Definition 2.6. Let $f \in R = \Bbbk[x_1,\dots,x_d]$ be a homogeneous polynomial where $\Bbbk$ is a field and assume that for some term order $\prec$ the leading term of f is a squarefree monomial. We define $\mathscr{C}_f$ to be the smallest set of ideals satisfying the following conditions:
(1) $(f) \in \mathscr{C}_f$.
(2) If $I \in \mathscr{C}_f$ then $(I : J) \in \mathscr{C}_f$ for every ideal $J \subseteq R$.
(3) If I and J are in $\mathscr{C}_f$ then also $I+J$ and $I \cap J$ must be in $\mathscr{C}_f$.
If I is an ideal in $\mathscr{C}_f$, we say that I is a *Knutson ideal associated to* f, or simply that it is *Knutson*.

If $\Bbbk$ has positive characteristic and $\deg(f) = d$, then f defines a splitting map on R (see [30, Lemma 4] for finite fields and [41] in general). Knutson ideals are compatibly split with respect to this map. Therefore, Knutson ideals define F-split rings in positive characteristic. In addition, Knutson ideals have squarefree initial ideals in arbitrary characteristic (see [30, Theorem 2] for finite fields and [41] in general) and consequently their extremal Betti numbers coincide with those of their initial ideals [12]. Furthermore, Gröbner bases of Knutson ideals are well-behaved with respect to sums as the union of Gröbner bases of Knutson ideals is a Gröbner basis of their sum. This quality makes computations of Gröbner bases for some Knutson ideals somewhat easier than for general ideals. All of these considerations make Knutson ideals relevant objects in computational algebra, combinatorial commutative algebra, and F-singularity theory.

Remark 2.7. Since every ideal of $\mathscr{C}_f$ is radical, condition Definition 2.6(2) can be replaced by:

$(2')$ If $I \in \mathscr{C}_f$ then $P \in \mathscr{C}_f$ for every $P \in \mathrm{Min}(I)$.

2.3. **Ladder determinantal ideals.** In this subsection we recall the definition of two-sided mixed ladder determinantal ideals. For this, we follow closely the notations and conventions form Gorla's work [20], where these ideals were introduced (see also [19]).

Definition 2.8. Let $X = (x_{i,j})$ be a generic matrix of size $k \times \ell$, i.e.,

$$X = \begin{bmatrix} x_{1,1} & \dots & x_{1,\ell} \\ \vdots & \vdots & \vdots \\ x_{k,1} & \dots & x_{k,\ell} \end{bmatrix}.$$

We say that a subset $L \subseteq X$ is a (*two sided*) *ladder* if it has the following property: if $x_{i,j}, x_{i',j'} \in L$ with $i \leqslant i'$ and $j \geqslant j'$, then $x_{u,v} \in L$ for any $i \leqslant u \leqslant i'$ and $j' \leqslant v \leqslant j$. We denote by

$$\{x_{1,a_1} = x_{b_1,a_1}, \ldots, x_{b_u,a_u} = x_{b_u,\ell}\} \text{ with } 1 = b_1 < \cdots < b_u \text{ and } a_1 < \cdots < a_u = \ell$$

the *upper-outside corners*, and by

$$\{x_{d_1,1} = x_{d_1,c_1}, \ldots, x_{d_v,c_v} = x_{k,c_v}\} \text{ with } d_1 \leqslant \cdots \leqslant d_v = k \text{ and } 1 = c_1 \leqslant \cdots \leqslant c_v$$

the *lower-outside corners* of L. Moreover, we assume all of the corners are distinct. If L has only one upper-outside corner we say it is *one-sided*. We also define the following subladders for all $j = 1, \ldots, v$

$$L_j = \{x_{b,a} \in L \mid 1 \leqslant b \leqslant d_j, c_j \leqslant a \leqslant l\}. \tag{2.1}$$

In Example 2.9 we illustrate all of these notations in a specific example.

Example 2.9. Let X be a generic matrix of size 10×10. In Figure 1 the shaded area represents the ladder L whose upper-outside corners are $\{x_{1,2}, x_{4,8}, x_{8,10}\}$ and lower-outside corners are $\{x_{3,1}, x_{6,1}, x_{8,6}, x_{10,8}\}$; note that the first two lower-outside corner belong to the same column.

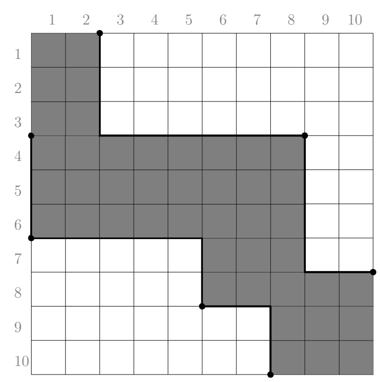

FIGURE 1. Ladder L

Since L has four lower-outside corners, it has four subladders L_1, L_2, L_3, L_4 as defined above. In Figure 2 we highlight these subladders.

Definition 2.10. Let $\Bbbk$ be a field and $L \subseteq X$ a ladder as in Definition 2.8. If t is a positive integer, we denote by $I_t(L)$ the ideal of $\Bbbk[L]$ generated by the t-minors of X that are completely contained in L. The ideal $I_t(L)$ is a *ladder determinantal ideal* of L. On the other hand, if $\mathbf{t} = (t_1, \ldots, t_v) \in \mathbb{Z}^v_{>0}$ we denote by $I_\mathbf{t}(L)$ the ideal $I_\mathbf{t}(L) = I_{t_1}(L_1) + \ldots + I_{t_v}(L_v)$. Such ideal is a *mixed ladder determinantal ideal* of L. Notice that, if $t = t_1 = \cdots = t_v$, then $I_\mathbf{t}(L) = I_t(L)$. The ideals $I_\mathbf{t}(L)$ and $I_t(L)$ are prime (see [20]).

Throughout, for every mixed ladder determinantal ideal we assume that all the entries of L are involved in some generator of $I_\mathbf{t}(L)$ and that $I_{t_i}(L_i) \not\subseteq I_{t_j}(L_j)$ for all $i \neq j$.

Example 2.11. Let L be as in Example 2.9. If $\mathbf{t} = (2,3,1,2)$, then $I_\mathbf{t}(L) = I_2(L_1) + I_3(L_2) + I_1(L_3) + I_2(L_4)$. If $\mathbf{t} = (2,2,2,2)$, then $I_\mathbf{t}(L) = I_2(L)$.

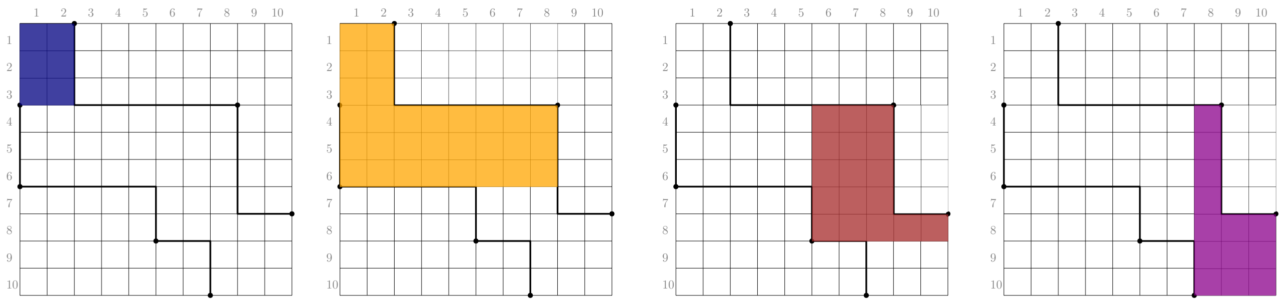

FIGURE 2. The subladders L_1, L_2, L_3, L_4 ordered from left to right

Notation 2.12. We summarize here the assumptions on the ladders treated in the present papers, which are the same as in Gorla's work [20].

(1) All the entries of L are involved in some minor, otherwise we delete the superfluous entries and consider the remaining ladder.
(2) No two consecutive corners coincide, and no two upper-outside corners belong to the same row or to the same column.
(3) $I_{t_i}(L_i) \not\subseteq I_{t_j}(L_j)$ for all $i \neq j$
(4) When we view L as a ladder inside the $k \times \ell$ matrix X we assume that there is no proper submatrix X' of X which contains L.

2.4. **Schubert determinantal ideals and poset ideals.** In this subsection we introduce further families of determinantal ideals treated in this paper. We begin by introducing some notations.

Notation 2.13. Let X be as in Definition 2.8. For any $1 \leqslant a < b \leqslant \ell$ and $1 \leqslant c < d \leqslant k$, we denote by $X_{[a,b]}^{[c,d]}$ the submatrix of X with column indices $a,\dots,b$ and row indices $c,\dots,d$. In the case $[c,d] = [1,k]$ (resp. $[a,b] = [1,\ell]$) we simply write $X_{[a,b]}$ (resp. $X^{[c,d]}$). We use $X_{a\times b}$ to denote the northwest submatrix $X_{[1,b]}^{[1,a]}$ and $X^{a\times b}$ to denote the southeast submatrix $X_{[\ell-b+1,\ell]}^{[k-a+1,k]}$. Moreover, for a set of indices $1 \leqslant i_1 < \cdots < i_r \leqslant k$ and $1 \leqslant j_1 < \cdots < j_r \leqslant \ell$ we denote by $[i_1 \cdots i_r \mid j_1 \cdots j_r]$ the determinant (r-minor) of the submatrix of X corresponding to rows $i_1,\dots,i_r$ and columns $j_1,\dots,j_r$.

Following [36], by the *antidiagonal term* of a minor $[i_1 \cdots i_r \mid j_1 \cdots j_r]$ we mean the product of the variables in the antidiagonal of the corresponding submatrix, namely $x_{i_1,j_r}x_{i_2,j_{r-1}}\cdots x_{i_r,j_1}$. Let $\Bbbk$ be an arbitrary field. A term order on the polynomial ring $\Bbbk[X]$ is *antidiagonal* if the initial term of any minor is its antidiagonal term. A typical example of an antidiagonal term order is the lexicographical one with variables ordered as

$$x_{1,\ell} > \cdots > x_{1,1} > x_{2,\ell} > \cdots > x_{2,1} > \cdots > x_{k,\ell} > \cdots > x_{k,1}.$$

Although it is not our intention to review the theory of *algebras with straightening law* in this paper (see [8, Chapter 3]), we wish to introduce a few concepts for the sake of clarity. The set Π of all the minors of the generic matrix X can be partially ordered by the relation

$$[a_1 \dots a_r | b_1 \dots b_r] \leqslant [c_1 \dots c_s | d_1 \dots d_s] \iff r \geqslant s, \;\; a_i \leqslant c_i \text{ and } b_i \leqslant d_i \;\text{ for all }\; i = 1, \dots, s.$$

The starting observation is that $\Bbbk[X]$ is generated as a $\Bbbk$-vector space by the set by

$$\{\pi_1 \cdots \pi_m \mid \;\pi_i \in \Pi \text{ for } i = 1, \dots, m \text{ and } \;\pi_1 \leqslant \cdots \leqslant \pi_m\}.$$

The elements of this $\Bbbk$-basis are called *standard monomials*. Although it may happen that the product of two standard monomials is not a standard monomial, such product will be uniquely writable as a $\Bbbk$-linear combination of standard monomials; this process is known in the literature as *straightening law*.

With the above notation, we are ready to define matrix Schubert varieties and its defining ideals (see [36, Chapter 15] for more information).

Definition 2.14. Let $\mathcal{M}_{k\times\ell}(\Bbbk)$ be the vector space of $k \times \ell$ matrices with entries in $\Bbbk$ and let $w = (w_{i,j}) \in \mathcal{M}_{k\times\ell}(\Bbbk)$ be a *partial permutation*, i.e., $w_{i,j} = 0$ for all i, j except for at most one entry equal to 1 in each row and column.

(1) The *matrix Schubert variety* $\overline{X_w}$ associated to w is given by

$$\overline{X_w} := \{Y \in \mathcal{M}_{k\times\ell} \mid \operatorname{rank}(Y_{r\times s}) \leqslant \operatorname{rank}(w_{r\times s}), \text{ for all } 1 \leqslant r \leqslant k, 1 \leqslant s \leqslant \ell\}.$$

(2) The defining ideal of $\overline{X_w}$ is the *Schubert determinantal ideal* I_w which is generated by the $\operatorname{rank}(w_{r\times s}) + 1$ minors of $X_{r\times s}$ for every $1 \leqslant r \leqslant k, 1 \leqslant s \leqslant \ell$.

Remark 2.15.

(1) Given $1 \leqslant t \leqslant \min\{k, \ell\}$, if $w \in \mathcal{M}_{k\times\ell}(\Bbbk)$ is the partial permutation defined by $w_{ii} = 1$ for $1 \leqslant i \leqslant t-1$, then I_w is the classical determinantal ideal of t-minors of X.
(2) Mixed ladder determinantal ideals of ladders with only one upper-outside corner coincide with the Schubert determinantal ideals of *vexillary permutations* [16, Proposition 9.6].
(3) For any partial permutation there are smaller sets of generators for I_w (see [36, Theorem 15.15], [17]).

We continue with the definition of the following class of ideals.

Definition 2.16. An *ideal* of Π is a subset $\Omega \subseteq \Pi$ such that

$$\text{for all } \omega \in \Omega \text{ and } \pi \in \Pi, \text{ if } \pi \leqslant \omega \text{ then } \pi \in \Omega.$$

For any Ω ideal of Π, we call the ideal generated by Ω in $\Bbbk[X]$ a *poset ideal*.

Remark 2.17. A particular type of ideals of Π, fixed $\delta \in \Pi$, are the sets

$$\Omega_\delta := \{\pi \in \Pi : \pi \not\geqslant \delta\}.$$

The poset ideals $\Omega_\delta \Bbbk[X]$ are the defining ideals of varieties originating as certain affine charts of Schubert varieties [38, p. 535] (see also [9, Chapter 5, §A]).

3. CHAMFERING LADDERS

In this section we define a combinatorial operation on ladders, which we call *chamfering*, and show some important consequences of it. This definition is inspired by the work of previous authors [10, 11, 19, 20].

Setup 3.1. We adopt the assumptions and notations from Definition 2.8, Definition 2.10, and Notation 2.12. In particular, $X = (x_{i,j})$ is a generic matrix of size $k \times \ell$ and $L \subseteq X$ is a ladder. Let $\Bbbk$ be a field. We denote by R the polynomial ring $R = \Bbbk[L]$. Given $\mathbf{t} = (t_1, \dots, t_v) \in \mathbb{Z}^v_{>0}$ we denote by $I_{\mathbf{t}}(L)$ the corresponding mixed ladder determinantal ideal.

Definition 3.2. Assume Setup 3.1. We say a ladder L is *non-degenerate* with respect to $\mathbf{t}$ if $I_{\mathbf{t}}(L)$ satisfies the assumptions in Notation 2.12. We call the quantity

$$\sum_{i<j} |t_i - t_j|$$

the *total width* of L with respect to $\mathbf{t}$. Let $\left\{x_{d_1,1} = x_{d_1,c_1}, \dots, x_{d_v,c_v} = x_{k,c_v}\right\}$ be the lower-outside corners of L. We say that a pair $(L', \mathbf{t}')$ with $L' \subset X$ a ladder and $\mathbf{t}' \in \mathbb{N}^d$ is the *chamfer* of $(L, \mathbf{t})$ at x_{d_j,c_j}, or that $(L', \mathbf{t}')$ is obtained from $(L, \mathbf{t})$) by *chamfering* x_{d_j,c_j}, if $\mathbf{t}' = (t_1, \dots, t_{j-1}, t_j - 1, t_{j+1}, \dots, t_v)$, and L' has the same upper-outside corners of L and lower-outside corners

$$\left\{x_{d_1,1} = x_{d_1,c_1}, \dots, x_{d_{j-1},c_{j-1}}, x_{d_j-1,c_j+1}, x_{d_{j+1},c_{j+1}}, \dots, x_{d_v,c_v} = x_{k,c_v}\right\}$$

for some x_{d_j,c_j} such that $d_j < d_{j-1}$ and $c_j > c_{j-1}$.

Proposition 3.3. *Assume Setup 3.1, and let L be a non-degenerate ladder with respect to $\mathbf{t}$. Then*

(1) *if L' is the chamfer of $(L, \mathbf{t})$ at x_{d_j,c_j}, then L' is non-degenerate.*
(2) *L is the chamfer of a non-degenerate ladder with strictly smaller total width.*

Proof. We note that condition (3) of Notation 2.12 holds if and only if $d_{j+1} - d_j > t_{j+1} - t_j$ and $c_{j+1} - c_j > t_j - t_{j+1}$, and one can readily check that it holds for L'. The fact that L' satisfies the other conditions of Notation 2.12 is an easy check. For the second claim, let j be such that $t_j = \min\{t_1, \dots, t_v\}$. Consider the ladder L'' with same upper-outside corners as L and lower outside corners

$$\left\{x_{d_1,1} = x_{d_1,c_1}, \dots, x_{d_{j-1},c_{j-1}}, x_{d_j+1,c_j-1}, x_{d_{j+1},c_{j+1}}, \dots, x_{d_v,c_v} = x_{k,c_v}\right\}.$$

We note that L'' is non-degenerate with respect to $\mathbf{t}'' = (t_1, \dots, t_{j-1}, t_j + 1, t_{j+1}, \dots, t_v)$, $(L, \mathbf{t})$ is the chamfer of $(L'', \mathbf{t}'')$ at x_{d_j+1,c_j-1}, and the total width of L'' is strictly smaller than L. $\square$

Corollary 3.4. *Assume Setup 3.1. If L is non-degenerate with respect to $\mathbf{t}$, then there exists a ladder L'' and $t, u \in \mathbb{N}$ such that L'' is non-degenerate with respect to $(t, \dots, t) \in \mathbb{N}^u$ and L is obtained from L'' by a finite sequence of chamferings.*

Proof. We proceed by descending induction on the total width of the ladder, making a repeated use of Proposition 3.3 (2). □

For an ideal I in a commutative ring with identity A, we denote the *symbolic Rees algebra* of I by $\mathscr{R}^s(I) = \oplus_{n\geqslant 0} I^{(n)}T^n \subseteq A[T]$, where T is a variable. Sometimes we write $\mathscr{R}^s_A(I) := \mathscr{R}^s(I)$ if the ambient ring needs to be specified. We denote by $\mathrm{gr}^s(I) = \oplus_{n\geqslant 0} I^{(n)}/I^{(n+1)}$ the *symbolic associated graded algebra* of I.

Theorem 3.5. *Assume Setup 3.1, and let L be a non-degenerate ladder with respect to* $\mathbf{t}$*. Then $\mathscr{R}^s(I_{\mathbf{t}}(L))$ and $\mathrm{gr}^s(I_{\mathbf{t}}(L))$ are Noetherian rings.*

Proof. It suffices to show the statement for $\mathscr{R}^s(I_{\mathbf{t}}(L))$. By Corollary 3.4 there exists a ladder L'' which is non-degenerate with respect to $(t,\dots,t) \in \mathbb{N}^u$, and L is obtained from L'' by a finite sequence of chamferings. Then by Gorla's work on mixed ladder determinantal ideals [20, Lemma 1.19] there are indeterminates $\{y_1,\dots,y_d\}$ of $R'' = \Bbbk[L'']$ and $c \leqslant d$ such that

$$I_{\mathbf{t}}(L)[y_1,\dots,y_d,y_1^{-1},\dots y_c^{-1}] \cong I_t(L'')[y_1^{-1},\dots y_c^{-1}].$$

From this isomorphism we conclude that for all $n \geqslant 1$

$$I_{\mathbf{t}}(L)^{(n)}[y_1,\dots,y_d,y_1^{-1},\dots y_c^{-1}] \cong I_t(L'')^{(n)}[y_1^{-1},\dots y_c^{-1}].$$

Since $\mathscr{R}^s_{R''}(I_t(L''))$ is Noetherian [7], so is $\mathscr{R}^s_R(I_{\mathbf{t}}(L))[y_1,\dots,y_d,y_1^{-1},\dots y_c^{-1}]$. As $\mathscr{R}^s_R(I_{\mathbf{t}}(L))$ is a direct summand of the latter, we conclude that it is itself Noetherian. □

Remark 3.6. Unmixed ladder determinantal ideals define F-rational singularities in characteristic $p \gg 0$, and rational singularities in characteristic zero, by work of Conca and Herzog [11]. It follows from Corollary 3.4 that this is also true for unmixed ladder determinantal ideals since being F-rational or having rational singularities localize, and because if S is a ring and x is an indeterminate one has that S is F-rational (resp. has rational singularities) if and only if $S[x]$ is F-rational (resp. has rational singularities).

4. Mixed ladder determinantal ideals

In this section we prove the main results of this paper regarding the F-singularities of ladder determinantal rings and their blowup algebras. Throughout we use the following setup:

Setup 4.1. We adopt Setup 3.1, and when the base field $\Bbbk$ has characteristic $p > 0$ we further assume that it is F-finite. For $r = 2,\dots,k+\ell$ we denote by $\mathscr{D}_r$ the antidiagonal at level r of X, namely,

$$\mathscr{D}_r = \{x_{i,j} \in X \mid i+j = r\} \subseteq X.$$

We define $L^\circ_{\mathbf{t}} \subseteq L$ to be the subladder with the same upper-outside corners as L, and with lower-outside corners $\{x_{d_j-t_j+1,c_j+t_j-1} \mid 1 \leqslant j \leqslant v\}$. Notice that $L^\circ_{\mathbf{t}} = \bigcup_{i=1}^{v}(L_j)^\circ_{t_j}$ (see (2.1)).

The following is the first main result of this section.

Theorem 4.2. *Assume Setup 4.1 with $\Bbbk$ of characteristic $p>0$. Then $I_{\mathbf{t}}(L)$ is symbolic F-split.*

Proof. Let $h=\operatorname{ht}(I_{\mathbf{t}}(L))$. It suffices to find an element $f\in I_{\mathbf{t}}(L)^{(h)}$ and a monomial order $\prec$ such that $\operatorname{in}_{\prec}(f)$ is squarefree [14, Lemma 6.2(1)]. Let

$$\mathscr{A}=\{2\leqslant r\leqslant k+\ell \mid \mathscr{D}_r\cap L^{\circ}_{\mathbf{t}}\neq\emptyset\},\tag{4.1}$$

and for every $r\in\mathscr{A}$ set

$$w_r=\max\{w\mid x_{w,r-w}\in\mathscr{D}_r\cap L^{\circ}_{\mathbf{t}}\}\qquad\text{and}\qquad p_r=\max\{j\mid x_{w_r,r-w_r}\in(L_j)^{\circ}_{t_j}\}.\tag{4.2}$$

Furthermore, set

$$a_r=\min\left\{i\mid x_{i,r-i}\in L_{p_r}\cap\mathscr{D}_r\right\}\qquad\text{and}\qquad b_r=\max\left\{i\mid x_{i,r-i}\in L_{p_r}\cap\mathscr{D}_r\right\}.\tag{4.3}$$

Let $\gamma_r=b_r-a_r+1$ and observe that the $\gamma_r\times\gamma_r$ matrix $Y_r=X^{[a_r,b_r]}_{[r-b_r,r-a_r]}$ is contained in L_{p_r} (see Notation 2.13). We note that

$$\det(Y_r)\in I_{\gamma_r}(Y_r)\subseteq I_{t_{p_r}}(Y_r)^{(\gamma_r-t_{p_r}+1)}\subseteq I_{t_{p_r}}(L_{p_r})^{(\gamma_r-t_{p_r}+1)}\subseteq I_{\mathbf{t}}(L)^{(\gamma_r-t_{p_r}+1)},\tag{4.4}$$

where the first inclusion follows from [9, Proposition 10.2].

Let $\mathscr{B}=\{r\in\mathscr{A}\mid \gamma_r-t_{p_r}+1\geqslant 0\}$ and set

$$f=\prod_{r\in\mathscr{B}}\det(Y_r)\tag{4.5}$$

We claim that $f\in I_{\mathbf{t}}(L)^{(h)}$. Observe that for every $r\in\mathscr{B}$, the integer $\max\{\gamma_r-t_{p_r}+1,0\}$ counts the number of elements on the main antidiagonal of Y_r that belong to $L^{\circ}_{\mathbf{t}}$. Moreover, every entry of $L^{\circ}_{\mathbf{t}}$ belongs to the main antidiagonal of Y_r for exactly one $r\in\mathscr{A}$. Thus, by [20, Theorem 1.15] we have

$$\sum_{r\in\mathscr{B}}(\gamma_r-t_{p_r}+1)=|L^{\circ}_{\mathbf{t}}|=h.\tag{4.6}$$

Therefore, (4.4) implies

$$f\in\prod_{r\in\mathscr{B}}I_{\mathbf{t}}(L)^{(\gamma_r-t_{p_r}+1)}\subseteq I_{\mathbf{t}}(L)^{(h)},$$

whence the claim follows. Finally, if $\prec$ is any antidiagonal term order, then $\operatorname{in}_{\prec}(f)$ is squarefree, completing the proof. □

As a consequence of Theorem 4.2 we obtain that symbolic blowup algebras of mixed ladder determinantal ideals, and hence also rings defined by such ideals, are F-pure.

Corollary 4.3. *Assume Setup 4.1 with $\Bbbk$ of characteristic $p>0$. Then the blowup algebras $\mathscr{R}^s(I_{\mathbf{t}}(L))$ and $\operatorname{gr}^s(I_{\mathbf{t}}(L))$ are F-pure. In particular, $R/I_{\mathbf{t}}(L)$ is F-pure.*

Proof. The first claim follows from Theorem 4.2 and properties of symbolic F-split ideals [14, Theorem 4.7]. The second follows the fact that $R/I_{\mathbf{t}}(L)$ is a direct summand of $\operatorname{gr}^s(I_{\mathbf{t}}(L))$. □

Let $S = \Bbbk[x_1,\dots,x_d]$ be a standard graded polynomial ring, and $\mathfrak{m} = (x_1,\dots,x_d)$. For a finitely generated $\mathbb{Z}$-graded S-module M we denote by $\operatorname{reg}(M)$ its *Castelnuovo-Mumford regularity*, i.e., $\sup\{j+i \mid H^i_{\mathfrak{m}}(M)_j \neq 0\}$.

Corollary 4.4. *Assume Setup 4.1 with $\Bbbk$ of characteristic $p>0$. Then, $\lim\limits_{n\to\infty} \frac{\operatorname{reg}\left(R/I_{\mathbf{t}}(L)^{(n)}\right)}{n}$ exists and the sequence $\{\operatorname{depth}(R/I_{\mathbf{t}}(L)^{(n)})\}_{n\in\mathbb{N}}$ is constant for $n \gg 0$, with value equal to $\min\{\operatorname{depth}(R/I_{\mathbf{t}}(L)^{(n)})\}_{n\in\mathbb{N}}$.*

Proof. The statements follow from Theorem 4.2 and properties of symbolic F-split ideals [14, Theorem 4.10], using the fact that $\mathscr{R}^s(I_{\mathbf{t}}(L))$ is Noetherian by Theorem 3.5. ☐

We also show that symbolic powers commute with taking initial ideals for unmixed ladder determinantal ideals.

Corollary 4.5. *Assume Setup 4.1 with $\Bbbk$ a perfect field of characteristic $p>0$, and let $I = I_{\mathbf{t}}(L)$. Let $\prec$ be an antidiagonal order, and assume that $\operatorname{in}_\prec(\mathscr{R}^s(I)) := \bigoplus_{n\geqslant 0} \operatorname{in}_\prec\left(I^{(n)}\right)$ is Noetherian. If $p \gg 0$, then we have*

$$\operatorname{in}_\prec\left(I^{(n)}\right) = (\operatorname{in}_\prec(I))^{(n)} \quad \textit{for every } n\in\mathbb{N}.$$

In particular, if $I = I_t(L)$ for some $t\in\mathbb{Z}_{>0}$, then the above equality holds for all $p > (\min\{k,\ell\}-t+1)!$.

Proof. This follows at once from Proposition 2.4. For the case $I = I_t(L)$, the bound for p follows from Remark 2.5 and the fact that $\operatorname{in}_\prec(\mathscr{R}^s(I))$ is generated as an R-algebra by forms of degree at most $\min\{k,\ell\}-t+1$ [7, Theorem 4.1]. ☐

The following technical lemma is needed for the proof of Theorem 4.7.

Lemma 4.6. *Let B be an F-finite ring of characteristic $p>0$, $J\subseteq B$ an ideal, and $A = B[x_1,\dots,x_d]$ a polynomial extension. Let $H = (x_1,\dots,x_d)$ and $I = JA+H$. Then $\mathscr{R}^s_A(I)$ is strongly F-regular if and only if $\mathscr{R}^s_B(J)$ is strongly F-regular.*

Proof. Since $H^{(j)} = H^j$ for all $j\in\mathbb{Z}_{>0}$, we have that $I^{(j)} = \sum_{i=0}^{j}(JA)^{(i)}H^{j-i}$ [23, Theorem 3.4]. Thus

$$\mathscr{R}^s_A(I) = \mathscr{R}^s_A(JA)[x_1T,\dots,x_dT] = \mathscr{R}^s_B(J)[x_1,\dots,x_d,x_1T,\dots,x_dT],$$

where the second equality holds as $(JA)^{(i)} = J^{(i)}A = J^{(i)}[x_1,\dots,x_d]$. Thus $\mathscr{R}^s_A(I)$ is isomorphic to $\mathscr{R}^s_B(J)[x_1Z,\dots,x_dZ,x_1T,\dots,x_dT]$, with Z another variable, and so isomorphic to the Segre product $\mathscr{R}^s_B(J)[x_1,\dots,x_d]\,\#\,\mathscr{R}^s_B(J)[Z,T]$.

Now, assume $\mathscr{R}^s_A(I)$ is strongly F-regular. Since $\mathscr{R}^s_B(J)$ is a direct summand of $\mathscr{R}^s_A(I)$, it is strongly F-regular [27, Theorem 3.1(e)]. Conversely, if $\mathscr{R}^s_B(J)$ is strongly F-regular then so is the polynomial extension $\mathscr{R}^s_B(J)[x_1,\dots,x_d,Z,T]$. Since $\mathscr{R}^s_A(I)$ is isomorphic to a direct summand of the latter, it is strongly F-regular as well, finishing the proof. ☐

The following is one of the main results of this paper. We show that the symbolic Rees algebra of any mixed ladder determinantal ideal is strongly F-regular.

Theorem 4.7. *Assume Setup 4.1 with $\Bbbk$ of characteristic $p>0$. Then $\mathscr{R}^s(I_{\mathbf{t}}(L))$ is strongly F-regular.*

Proof. We proceed by induction on $d=\dim(R)$, i.e., on the number of entries of. The base case $d=1$ is trivial.

It $t_v=1$ then we can write $I_{\mathbf{t}}(L)=I_{\mathbf{t}'}(L')+I_1(L_v)$ where L' is a proper subladder of L and $\mathbf{t}'=(t_1,\dots,t_{v-1})$. Let $R'=\Bbbk[L']$. Since $\dim(R')<\dim(R)$ by induction hypothesis we have that $\mathscr{R}^s_{R'}(I_{\mathbf{t}'}(L'))$ is strongly F-regular. It follows that $\mathscr{R}^s_R(I_{\mathbf{t}}(L))$ is strongly F-regular by Lemma 4.6.

Now assume $t_v>1$ and consider the smallest $\alpha\in\{1,\dots,v\}$ for which (k,c_α) is a lower-outside corner of L. We observe that such a α exists since (k,c_v) is always a lower-outside corner by our assumptions (see Notation 2.12). Let $\mathscr{A},p_r,a_r,b_r$ be as in (4.1), (4.2), and (4.3). For every $r\in\mathscr{A}$ set $\gamma_r=b_r-a_r+1$ and $Y_r=X^{[a_r,b_r]}_{[r-b_r,r-a_r]}$. By (4.4) we have $\det(Y_r)\in I_{\mathbf{t}}(L)^{(\gamma_r-t_{p_r}+1)}$. Let $\mathscr{B}=\{r\in\mathscr{A}\mid \gamma_r-t_{p_r}+1\geqslant 0\}$ and note that $\beta:=k+\gamma_\alpha\in\mathscr{B}$, $b_\beta=k$, and $p_\beta=v$. Set $Y=X^{[a_\beta,k-1]}_{[c_\alpha+1,\beta-a_\beta]}$ and observe that $\det(Y)\in I_{\gamma_\beta-1}(Y)\subseteq I_{t_v}(Y)^{(\gamma_\beta-t_v)}\subseteq I_{\mathbf{t}}(L)^{(\gamma_\beta-t_v)}$ [9, Proposition 10.2]. Set

$$\mathscr{B}'=\mathscr{B}\smallsetminus\{\beta\}\qquad\text{and}\qquad g=\det(Y)\prod_{r\in\mathscr{B}'}\det(Y_r).$$

We claim that $g\in I_{\mathbf{t}}(L)^{(h-1)}$, where $h=\operatorname{ht}(I_{\mathbf{t}}(L))$. For every $r\in\mathscr{B}$, the integer $\max\{\gamma_r-t_{p_r}+1,0\}$ counts the number of elements on the main antidiagonal of Y_r that belong to $L^\circ_{\mathbf{t}}$. By (4.6) we have

$$(\gamma_\beta-t_v)+\sum_{r\in\mathscr{B}'}(\gamma_r-t_{p_r}+1)=\sum_{r\in\mathscr{B}}(\gamma_r-t_{p_r}+1)=h-1.$$

Therefore $g\in I_{\mathbf{t}}(L)^{(\gamma_\beta-t_v)}\prod_{r\in\mathscr{B}'}I_{\mathbf{t}}(L)^{(\gamma_r-t_{p_r}+1)}\subseteq I_{\mathbf{t}}(L)^{(h-1)}$, which proves the claim.

Now let $\prec$ be any antidiagonal term order and note that $\operatorname{in}_\prec(g)$ is a squarefree monomial not divisible by x_{k,c_α}. Moreover, we have $g^{p-1}\in\left(I_{\mathbf{t}}(L)^{(n)}\right)^{[p]}:I_{\mathbf{t}}(L)^{(np)}$ for every $n\in\mathbb{N}$ [21, Lemma 2.6]. Thus there exists an $\mathscr{R}^s_R(I_{\mathbf{t}}(L))$-homomorphism $\Psi:\mathscr{R}^s_R(I_{\mathbf{t}}(L))^{1/p}\to\mathscr{R}^s_R(I_{\mathbf{t}}(L))$ such that $\Psi\left(x^{1/p}_{k,c_\alpha}\right)=1$ (cf. proof of [14, Theorem 6.7]).

Since $\mathscr{R}^s(I_{\mathbf{t}}(L))$ is Noetherian by Theorem 3.5, and $\Bbbk$ is F-finite, we have that $\mathscr{R}^s(I_{\mathbf{t}}(L))$ is F-finite. By [27, Theorem 3.3] to finish the proof it suffices to show that $\mathscr{R}^s_R(I_{\mathbf{t}}(L))_{x_{k,c_\alpha}}$ is strongly F-regular. Let L'' be the ladder obtained from L by deleting the entries

$$E=\{x_{d_{\alpha-1}+1,c_\alpha},x_{d_{\alpha-1}+2,c_\alpha},\dots,x_{k,c_\alpha},x_{k,c_\alpha+1},\dots,x_{k,c_{\alpha+1}-1}\}$$

from L. Set $R''=\Bbbk[L'']$ and $S=R''[E]_{x_{k,c_\alpha}}$. There is an isomorphism

$$\theta:R_{x_{k,c_\alpha}}\to S\quad\text{ such that }\quad\theta\left(I_{\mathbf{t}}(L)_{x_{k,c_\alpha}}\right)=I_{\mathbf{t}''}(L'')S,$$

where $\mathbf{t}''=(t_1,\dots,t_{\alpha-1},t_\alpha-1,t_{\alpha+1},t_{v-1},t_v)$ and $I_{\mathbf{t}''}(L'')$ is computed in R'' [20, Lemma 1.19]. One can check that $\theta\left(I_{\mathbf{t}}(L)^{(n)}_{x_{k,c_\alpha}}\right)=I_{\mathbf{t}''}(L'')^{(n)}S$ for all $n\geqslant 1$ (see [9, Lemma 10.1] for an analogous procedure). By induction hypothesis $\mathscr{R}^s_{R''}(I_{\mathbf{t}''}(L''))$ is strongly F-regular, and therefore $\mathscr{R}^s_{R''}(I_{\mathbf{t}''}(L''))\otimes_{R''}S$

is strongly F-regular. Therefore we conclude via the isomorphism θ that $\mathscr{R}^s_R(I_{\mathbf{t}}(L)) \otimes_R R_{x_{k,c\alpha}} \cong \mathscr{R}^s_R(I_{\mathbf{t}}(L))_{x_{k,c\alpha}}$ is strongly F-regular, finishing the proof. $\square$

The rest of the this section is devoted to proving the following theorem which states that unmixed determinantal ideals of two-sided ladders are Knutson ideals. Following Notation 2.13, for a ladder $L \subset X$ we denote by $L_{[a,b]} = X_{[a,b]} \cap L$ the subladder corresponding to the consecutive columns from a to b. Likewise, $L^{[c,d]} = X^{[c,d]} \cap L$ denotes the subladder corresponding to the consecutive rows from c to d.

Theorem 4.8. *Assume Setup 4.1. For $t \in \mathbb{Z}_{>0}$ we let $I_t(L) \subseteq R = \Bbbk[L]$ be the corresponding unmixed ladder determinantal ideal. Let f be as in (4.5) with $t_1 = \cdots = t_v = t$. Then*

$$I_t\left(L_{[a,b]}\right) \in \mathscr{C}_f \qquad \text{for every} \qquad 1 \leqslant a < b \leqslant \ell \quad \text{such that} \quad I_t\left(L_{[a,b]}\right) \neq 0, \qquad \text{and}$$

$$I_t\left(L^{[c,d]}\right) \in \mathscr{C}_f \qquad \text{for every} \qquad 1 \leqslant c < d \leqslant k \quad \text{such that} \quad I_t\left(L^{[c,d]}\right) \neq 0.$$

In particular, then $I_t(L) \in \mathscr{C}_f$ whenever $I_t(L)$ is not zero.

A first step towards the proof of Theorem 4.8 is the following lemma where we show that the ideal generated by the t-minors of a subladder corresponding to t consecutive columns or rows is in $\mathscr{C}_f$.

Lemma 4.9. *Under the assumptions and notations as in Theorem 4.8 we have*

$$I_t\left(L_{[j,j+t-1]}\right) \in \mathscr{C}_f \qquad \text{for every} \quad 1 \leqslant j \leqslant \ell - t + 1 \quad \text{such that} \quad I_t\left(L_{[j,j+t-1]}\right) \neq 0, \qquad \text{and}$$

$$I_t\left(L^{[i,i+t-1]}\right) \in \mathscr{C}_f \qquad \text{for every} \qquad 1 \leqslant i \leqslant k - t + 1 \quad \text{such that} \quad I_t\left(L^{[i,i+t-1]}\right) \neq 0.$$

Proof. Fix $1 \leqslant j \leqslant \ell - t + 1$ such that $I_t\left(L_{[j,j+t-1]}\right) \neq 0$. Set $L' = L_{[j,j+t-1]}$ and $\mathscr{E} = \{2 \leqslant r \leqslant k+\ell \mid |\mathscr{D}_r \cap L'| = t\}$. Notice that $\mathscr{E} \neq 0$, and for every $r \in \mathscr{E}$ there is only one element on the main antidiagonal of Y_r that belongs to $(L')^\circ_t$ (see sentence after (4.3)). Then by [20, Theorem 1.15] it follows that $\operatorname{ht}(I_t(L')) = |(L)^\circ_t| = |\mathscr{E}|$. Moreover

$$H := (\{\det(Y_r) \mid r \in \mathscr{E}\}) \subseteq I_t(L').$$

Notice H is a complete intersection with $\operatorname{ht}(H) = |\mathscr{E}|$ and $H \in \mathscr{C}_f$ as it is generated by some of the factors of f. Thus $I_t(L')$ is a minimal prime of of H and so $I_t(L') \in \mathscr{C}_f$.

Likewise, one can prove that $I_t\left(L^{[i,i+t-1]}\right)$ is either zero or in $\mathscr{C}_f$ for every i. $\square$

With Lemma 4.9 in hand we can prove Theorem 4.8.

Proof of Theorem 4.8. We only show $I_t\left(L_{[a,b]}\right) \in \mathscr{C}_f$ whenever $I_t\left(L_{[a,b]}\right) \neq 0$, as the proof for $I_t\left(L^{[c,d]}\right)$ is identical.

We proceed by induction on $\delta \in \mathbb{N}$ to show that $I_t\left(L_{[j,j+t-1+\delta]}\right)$ is either 0 or in $\mathscr{C}_f$ for every $1 \leqslant j \leqslant \ell - t + 1 - \delta$, the base case $\delta = 0$ being Lemma 4.9.

Fix $1 \leqslant j \leqslant \ell - t - \delta$. By induction hypothesis the ideals $I_t\left(L_{[j,j+t-1+\delta]}\right)$ and $I_t\left(L_{[j+1,j+t+\delta]}\right)$ are either zero or in $\mathscr{C}_f$. Therefore so is their sum. Assume that the sum is not zero.

We claim that

$$I_t\left(L_{[j,j+t-1+\delta]}\right) + I_t\left(L_{[j+1,j+t+\delta]}\right) = I_t\left(L_{[j,j+t+\delta]}\right) \cap I_{t-1}\left(L_{[j+1,j+t-1+\delta]}\right), \tag{4.7}$$

if $t > 1$, and

$$I_t\left(L_{[j,j+t-1+\delta]}\right) + I_t\left(L_{[j+1,j+t+\delta]}\right) = I_t\left(L_{[j,j+t+\delta]}\right), \tag{4.8}$$

otherwise.

Since (4.8) is clear, we focus on proving (4.7). So, we may assume $t > 1$. To simplify the notation we set

$$L' := L_{[j,j+t-1+\delta]} \qquad \text{and} \qquad L'' := L_{[j+1,j+t+\delta]}.$$

From the proof of [42, Theorem 2.1] it follows that

$$I_t\left(X_{[j,j+t-1+\delta]}\right) + I_t\left(X_{[j+1,j+t+\delta]}\right) = I_t\left(X_{[j,j+t+\delta]}\right) \cap I_{t-1}\left(X_{[j+1,j+t-1+\delta]}\right).$$

Thus after contracting these ideals to $\Bbbk[L' \cup L'']$ we obtain

$$\left(I_t\left(X_{[j,i+t-1+\delta]}\right) + I_t\left(X_{[j+1,j+t+\delta]}\right)\right) \cap \Bbbk[L' \cup L''] = I_t(L') + I_t(L''), \qquad \text{and}$$

$$\left(I_t\left(X_{[j,j+t+\delta]}\right) \cap I_{t-1}\left(X_{[j+1,j+t-1+\delta]}\right)\right) \cap \Bbbk[L' \cup L''] = I_t(L' \cup L'') \cap I_{t-1}(L' \cap L''),$$

where the first equality follows from [42, Lemma 2.2], [30, Corollary 2], and [20, Lemma 1.8], whereas the second one is clear. This proves the claim.

Hence the ideal $I_t(L_{[j,j+t+\delta]}) = I_t(L' \cup L'')$ is a minimal prime of $I_t(L') + I_t(L'') \in \mathscr{C}_f$. We conclude that $I_t(L_{[j,j+t+\delta]})$ is either 0 or in $\mathscr{C}_f$ for every $1 \leqslant j \leqslant \ell - t$, finishing the proof. $\square$

5. Knutson Schubert determinantal and poset ideals

In this section, we show that Schubert determinantal ideals and poset determinantal ideals are Knutson ideals. We remark that the statement for Schubert determinantal ideals already appeared in Knutson's work [30].

Throughout this section we follow the notations introduced in §2.4. In particular, $X = (x_{i,j})$ is a generic matrix of size $k \times \ell$. We also assume the following setup.

Setup 5.1. We adopt Notation 2.13. Let $R = \Bbbk[X]$ be the polynomial ring in the variables in X, where $\Bbbk$ is a an arbitrary field. Moreover, we equip R with an antidiagonal term order.

We continue with the definition of the following polynomial which plays an essential role in the results in this section. This polynomial is inspired by the results by the fourth-named author [42].

Definition 5.2. Assume Setup 5.1. We define $f(X) \in \Bbbk[X]$ as:

$$f(X) = \prod_{j=0}^{k-2}\left[\det\left(X_{[1,j+1]}^{[1,j+1]}\right)\det\left(X_{[\ell-j,\ell]}^{[k-j,k]}\right)\right]\prod_{j=1}^{\ell-k+1}\left(\det\left(X_{[j,k+j-1]}\right)\right).$$

We note that:

$$\operatorname{in}(f(X)) = \prod_{1\leqslant i\leqslant k, 1\leqslant j\leqslant \ell} x_{i,j}.$$

The following theorem shows that determinantal ideals of every northwest and southeast submatrix of X is Knutson associated to $f(X)$. This allows us to show Schubert determinantal and poset ideals are Knutson

Theorem 5.3. *Assume Setup 5.1 and let $f := f(X)$ be as in Definition 5.2. Then for every $1 \leqslant r \leqslant k$, $1 \leqslant s \leqslant \ell$, and $t \leqslant \min\{r,s\}$, we have $I_t(X_{r\times s}) \in \mathscr{C}_f$ and $I_t(X^{r\times s}) \in \mathscr{C}_f$.*

Proof. If $r = k$, or $s = \ell$, then $I_t(X_{r\times s}) \in \mathscr{C}_f$ [42, Lemma 2.2]. Hence, we can assume $r \neq k$ and $s \neq \ell$. We set $m := \min\{r,s\}$ and $M := \max\{r,s\}$. We proceed by induction on $\delta := m-t$.

Base case: Assume $\delta = 0$. We define,

$$D_1 := (\det(X_{m\times m}), \ldots, \det(X_{M\times M})) \text{ if } M \leqslant k, \text{and}$$

$$D_1 := \left(\det(X_{m\times m}), \ldots, \det(X_{k\times k}), \det\left(X_{[2,k+1]}\right), \ldots, \det\left(X_{[M-k+1,M]}\right)\right) \text{ otherwise.}$$

The ideal D_1 is a complete intersection of height $M-m+1 = \operatorname{height}(I_m(X_{r\times s}))$ and it is contained in $I_m(X_{r\times s})$, therefore $I_m(X_{r\times s}) \in \operatorname{Min}(D_1)$. Moreover, $D_1 \in \mathscr{C}_f$ because it is generated by factors of f and so it is a sum of minimal primes of (f). Thus, $I_m(X_{r\times s}) \in \mathscr{C}_f$ for every r and s.

Induction step: Assume $\delta \geqslant 1$. We have the inclusion

$$I_t(X_{r\times s}) \supseteq D_2 := I_t(X_{r-1\times s}) + I_t(X_{r\times s-1}) + I_{t+2}(X_{r+1\times s+1}). \tag{5.1}$$

The ideal $J := I_t(X_{r-1\times s}) + I_t(X_{r\times s-1})$ is ladder determinantal (see §2.3), so it is prime of height $(r-t+1)(s-t+1)-1$ [20]. Moreover, J does not contain $I_{t+2}(X_{r+1\times s+1})$, then

$$(r-t+1)(s-t+1) = \operatorname{height}(I_t(X_{r\times s})) \geqslant \operatorname{height}(D_2) \geqslant \operatorname{height}(J)+1 \geqslant (r-t+1)(s-t+1),$$

this implies

$$I_t(X_{r\times s}) \in \operatorname{Min}(D_2). \tag{5.2}$$

Now, assume $s \geqslant r$, i.e,, $m = r$. By the case $\delta-1$ of the induction $I_t(X_{r-1\times s})$ and $I_{t+2}(X_{r+1\times s+1})$ belong to $\mathscr{C}_f$. If $r = s$, we have $I_t(X_{r\times s-1}) \in \mathscr{C}_f$ again by the case $\delta-1$ of the induction, and then $I_t(X_{r\times s}) \in \mathscr{C}_f$ by (5.1) and (5.2). Now, using (5.1) and (5.2) repeatedly we conclude $I_t(X_{r\times s}) \in \mathscr{C}_f$ for any $s \geqslant r$. The case $r \geqslant s$ follows similarly by reversing the roles of r and s. Thus, $I_t(X_{r\times s}) \in \mathscr{C}_f$ as we wanted to show.

Likewise one shows that $I_t(X^{r\times s}) \in \mathscr{C}_f$, finishing the proof. $\square$

As a consequence, we conclude that Schubert determinantal ideals are Knutson ideals for the same choice of f, which recovers a result by Knutson [30].

Corollary 5.4. *Assume Setup 5.1 and let $f := f(X)$ be as in Definition 5.2. Let $w \in \mathcal{M}_{k\times\ell}(\Bbbk)$ be a partial permutation and let I_w be its corresponding Schubert determinantal ideal. Then $I_w \in \mathscr{C}_f$. In particular, $\operatorname{in}_\prec(I_w)$ is a squarefree monomial ideal for any antidiagonal term order $\prec$ and Schubert determinantal ideals define F-pure rings in positive characteristic.*

Proof. Notice that I_w can be written as a finite sum of ideals of the form $I_t(X_{r\times s})$. Thus, the result follows from Theorem 5.3. $\square$

Remark 5.5. *Alternating sign matrix varieties* and their defining ideals are generalizations of matrix Schubert varieties (see [29] for more details). Theorem 5.3 applies to these ideals to show that they belong to $\mathscr{C}_f$. In particular, their initial ideals with respect to any antidiagonal term order are squarefree and alternating sign matrix ideals define F-pure rings in positive characteristic.

The next result about poset ideals generalizes [3, Remark 79] from maximal minors to minors of any size.

Corollary 5.6. *Assume Setup 5.1 and let $f := f(X)$ be as in Definition 5.2. For any ideal $\Omega \subseteq \Pi$, the poset ideal $\Omega\Bbbk[X]$ is in $\mathscr{C}_f$. In particular, $\operatorname{in}_\prec(\Omega\Bbbk[X])$ is a squarefree monomial ideal for any antidiagonal term order $\prec$ and $\Bbbk[X]/\Omega\Bbbk[X]$ is F-pure in positive characteristic.*

Proof. Given $\delta \in \Pi$ we set $\Omega_\delta = \{\pi \in \Pi : \pi \not\geqslant \delta\} \subseteq \Pi$. Notice that Ω_δ is an ideal of Π, and we have the equality $\Omega = \bigcap_{\delta\in\min\{\Pi\setminus\Omega\}} \Omega_\delta$. The, we also have $\Omega\Bbbk[X] = \bigcap_{\delta\in\min\{\Pi\setminus\Omega\}}(\Omega_\delta\Bbbk[X])$ [9, Proposition 5.2]. If $\delta = [i_1 \dots i_r | j_1 \dots j_r]$, then

$$\Omega_\delta\Bbbk[X] = \sum_{t=1}^{r}\left(I_t\left(X_{[1,j_t-1]}\right) + I_t\left(X^{[1,i_t-1]}\right)\right) + I_{r+1}(X).$$

Therefore, the ideals $\Omega_\delta\Bbbk[X]$ belong to $\mathscr{C}_f$ by Theorem 5.3. Thus, their intersection $\Omega\Bbbk[X]$ belongs to $\mathscr{C}_f$ as well. Finally, the conclusion follows as $\Omega\Bbbk[X]$ is the intersection of ideals in $\mathscr{C}_f$. $\square$

Remark 5.7. There are graded algebras with straightening law over fields of positive characteristic that are not F-pure [32, Remark 5.2], which gives a counterexample to a conjecture stated in page 245 of [15]. However, Corollary 5.6 confirms the conjecture in many situations.

Remark 5.8. The fact that for any antidiagonal term order $\prec$ the ideals $\operatorname{in}_\prec(I_w)$ [31, Theorem B] and $\operatorname{in}_\prec(\Omega\Bbbk[X])$ [25, Theorem 2.5] are squarefree was already known.

We introduce a new notion generalizing that of poset ideal. Given $1 \leqslant r \leqslant k, 1 \leqslant s \leqslant l$, we set $\Pi_{r,s}$ to be the set of all minors of $X_{r\times s}$.

Definition 5.9. A *generalized ideal* of Π is a subset $\Omega \subseteq \Pi$ such that

$$\text{for all } [i_1 \cdots i_r | j_1 \cdots j_r] \in \Omega \text{ and } \pi \in \Pi_{i_r,j_r}, \text{ if } \pi \leqslant [i_1 \cdots i_r | j_1 \cdots j_r] \text{ then } \pi \in \Omega.$$

Note that any ideal of Π is a generalized ideal, but there exist generalized ideals which are not ideals themselves, like the sets $\Pi_{r,s}$.

Since a generalized poset ideal is the sum of ideals of form $\Pi_{r,s}$ (where r,s vary), the proof of Corollary 5.6 can be adapted to show the following result.

Corollary 5.10. *Assume Setup 5.1 and let $f := f(X)$ be as in Definition 5.2. For any generalized ideal $\Omega \subseteq \Pi$, the generalized poset ideal $\Omega\Bbbk[X]$ is in $\mathscr{C}_f$. In particular,* $\mathrm{in}_{\prec}(\Omega\Bbbk[X])$ *is a squarefree monomial ideal for any antidiagonal term order $\prec$, and $\Bbbk[X]/\Omega\Bbbk[X]$ is F-pure in positive characteristic.*

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