Dual graph of projective curves

Commutative Algebra TOwards Applications

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Setting the table

- $\Bbbk$ algebraically closed field;
- $C = C_1 \cup \ldots \cup C_s \subseteq \mathbb{P}^n$ projective curve with primary components projective curves $C_i \subseteq \mathbb{P}^n$;
- $I_C = I_{C_1} \cap \ldots \cap I_{C_s} \subset S = \Bbbk[X_0, \ldots, X_n]$ irredundant primary decomposition of the ideal of definition of $C \subset \mathbb{P}^n$;
- $G(C)$ graph on $\{1, \ldots, s\}$ with edges $\{i, j\}$ iff $C_i \cap C_j \neq \emptyset$. $G(C)$ is called the **dual graph** of C (or of I_C).

Remarks

$G(C) = G(C_{\text{red}})$. Also, C is connected $\Leftrightarrow$ $G(C)$ is connected.

Hartshorne

If $C \subseteq \mathbb{P}^n$ is a **complete intersection** (i.e. $I_C = (f_1, \ldots, f_{n-1})$), then $G(C)$ is connected.

We say that $C \subset \mathbb{P}^n$ is a **set-theoretic complete intersection** if $C_{\text{red}} = C'_{\text{red}}$ where $C' \subset \mathbb{P}^n$ is a ci (i.e. $C \subset \mathbb{P}^n$ is a set-ci if there are homogeneous $f_1, \ldots, f_{n-1} \in S$ s.t. $\sqrt{I_C} = \sqrt{(f_1, \ldots, f_{n-1})}$). If $C \subset \mathbb{P}^n$ is a set-ci, then C is connected, and whether the converse holds is an open problem since the seventies:

Problem

Is any connected curve $C \subset \mathbb{P}^n$ a set-ci?

The above problem is wide open already if $n = 3$, even for "innocent looking" examples such as

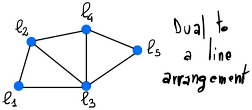
$$C = \{[x^4, x^3y, xy^3, y^4] : [x, y] \in \mathbb{P}^1\} \subset \mathbb{P}^3, \quad \Bbbk = \mathbb{C}.$$

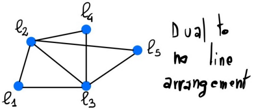
Mohan Kumar

If $C \subset \mathbb{P}^3$ is a connected union of lines (so C_i is a line for all i), then C is a set-ci.

Motivations and basic results

Not any connected graph is the dual graph of an union of lines though, for example:





However the second graph G is the dual graph of a projective curve $C \subset \mathbb{P}^3$: take $C_0 = \cup_{i=1}^{5} \ell_i \subset \mathbb{P}^2$ where the ℓ_i's are generic lines in $\mathbb{P}^2$. Note that $G(C_0)$ is the complete graph on 5 vertices. Consider the set X of points corresponding to the non-edges of G, i.e.:

$$X = \{\ell_1 \cap \ell_4,\ \ell_1 \cap \ell_5,\ \ell_4 \cap \ell_5\}.$$

Let $S \subset \mathbb{P}^n$ be the blow-up of $\mathbb{P}^2$ along X, and $C_1 \subset S$ the strict transform of C_0. $G(C_1) = G$ by construction, and since C_1 has only planar singularities it can be embedded in $\mathbb{P}^3$:

$$C_1 \cong C \subset \mathbb{P}^3.$$

This reasoning gives:

Benedetti-Bolognese-_

Given a connected graph G there is $C \subset \mathbb{P}^3$ such that $G = G(C)$.

Then the following is a sub-problem of the previous one:

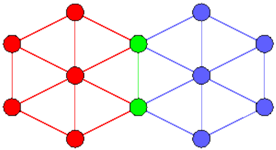
Sub-problem

Given a connected graph G, is there a complete intersection $C \subset \mathbb{P}^3$ such that $G = G(C)$?

Notions from graph theory

Given a simple graph G on s vertices and an integer r, we say that G is r-**connected** if the removal of less than $\min\{r, s-1\}$ vertices of G does not disconnect it. The **valency** of a vertex v of G is:

$$\delta(v) = |\{w : \{v, w\} \text{ is an edge of } G\}|.$$



- 2-connected, not 3-connected.
- $\delta(\bullet) = 5$.
- $\delta(\text{inner}) = \delta(\text{inner}) = 6$.
- $\delta(\text{boundary}) = \delta(\text{boundary}) = 3$.

Remark

The usual definition requires $r < s$. According to our definition the complete graph on s vertices is r-connected for all $r \in \mathbb{Z}$.

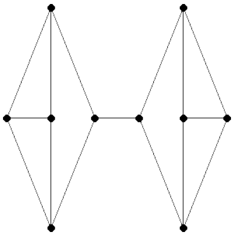
Remark

(i) G is 1-connected $\Leftrightarrow$ G is connected.

(ii) G is r-connected $\Rightarrow$ G is r'-connected for all $r' < r$.

(iii) G is r-connected on s vertices $\Rightarrow$ $\delta(v) \geq \min\{r, s-1\}$ for all vertices v of G.

G is said to be r-**regular** if $\delta(v) = r$ for any vertex v.



3-regular, connected, not 2-connected.

Given a homogeneous $I \subseteq S = \Bbbk[X_0, \ldots, X_n]$, if $\mathfrak{m} = (X_0, \ldots, X_n)$:

$$\operatorname{reg}(S/I) = \max\{i + j : H^i_{\mathfrak{m}}(S/I)_j \neq 0\}.$$

For $C \subset \mathbb{P}^n$, we let $\operatorname{reg}(C) = \operatorname{reg}(S/I_C) + 1$.

Definition

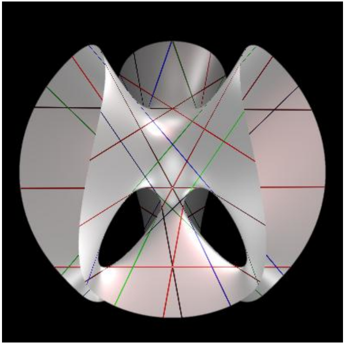
We say that $C \subset \mathbb{P}^3$ is **a complete intersection of type** (d, e) if $I_C = (f, g)$ with $\deg(f) = d$ and $\deg(g) = e$.

Remark

If $C \subset \mathbb{P}^3$ is a complete intersection of type (d, e), then $\operatorname{reg}(C) = d + e - 1$.

Let $Z \subseteq \mathbb{P}^3$ be a smooth cubic, and $C = \bigcup_{i=1}^{27} C_i$ be the union of all the lines on Z. Below is a representation of the Clebsch's cubic:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = (x_0 + x_1 + x_2 + x_3)^3.$$



Example: 27 lines

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\bigcup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(C)$:

- let i be the vertex corresponding to E_i;
- let ij be the vertex corresponding to the strict transform of the line passing through P_i and P_j;
- let i be the vertex corresponding to the strict transform of the conic avoiding P_i;

One easily checks that:

- $\{i, jk\}$ is an edge of $G(C) \Leftrightarrow i \in \{j, k\}$;
- $\{i, j\}$ is an edge of $G(C) \Leftrightarrow i \neq j$;
- $\{ij, k\}$ is an edge of $G(C) \Leftrightarrow k \in \{i, j\}$;
- $\{ij, hk\}$ is an edge of $G(C) \Leftrightarrow \{i, j\} \cap \{h, k\} = \emptyset$;
- $\{i, j\}$ and $\{i, j\}$ are never edges of $G(C)$.

Example: 27 lines

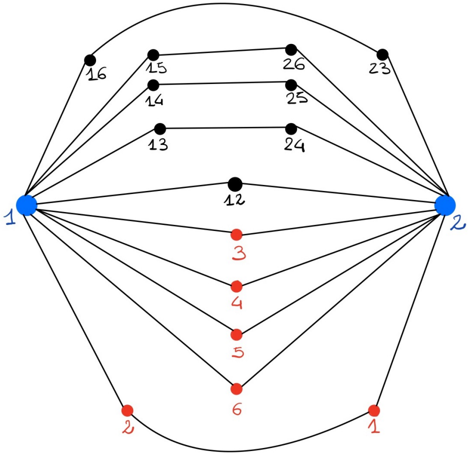
As it turns out $C \subseteq \mathbb{P}^3$ is a complete intersection of the cubic Z and a union of 9 planes, hence of type $(3, 9)$. One can check that:

- $\operatorname{reg} C - 1 = 10$.
- $G(C)$ is 10-connected.
- $G(C)$ is 10-regular.

To check that $G(C)$ is 10-connected is convenient to use a theorem of Menger: A simple graph G on s-vertices is r-regular for a given $r < s$ if and only if for all pair of distinct vertices v and w there are at least r vertex-disjoint paths connecting them.

Example: 27 lines

Paths from 1 to 2.



Theorem (Benedetti-Bolognese,_)

Let $C = C_1 \cup \ldots \cup C_s \subseteq \mathbb{P}^3$ a complete intersection of type (d, e) with primary components C_i. Let $r = \max\{\text{reg}(C_i) : i = 1, \ldots, s\}$. Then $G(C)$ is $\lfloor \frac{d+e+r-3}{r} \rfloor$-connected. If C is furthermore reduced, we can replace r with $r' = \max\{\deg(C_i) : i = 1, \ldots, s\}$.

Corollary

Let $C = C_1 \cup \ldots \cup C_s \subseteq \mathbb{P}^3$ a complete intersection of type (d, e) with primary components C_i. Suppose that $\text{reg}(C_i) \leq d + e - 3$ for all $i = 1 \ldots, s$. Then $G(C)$ is 2-connected.

Corollary on line arrangements (Benedetti,_)

Let $C = C_1 \cup \ldots \cup C_s \subseteq \mathbb{P}^3$ a complete intersection of type (d, e) with as primary components **lines** $C_i \subset \mathbb{P}^3$ (in particular C is reduced). Then $G(C)$ is $(d + e - 2)$-connected.

The proof uses:

- Liaison theory.
- $H^1_{\mathfrak{m}}(S/I_C)_0$ implies C is connected.
- $\mathrm{reg}(C) \leq \mathrm{reg}(C_1) + \ldots + \mathrm{reg}(C_s)$ (Caviglia).

Indeed, we prove a similar result for $X \subset \mathbb{P}^n$ with S/I_X Gorenstein.

With **Hongmiao Yu** we extended liaison theory via Gorenstein varieties to liaison theory via quasi-Gorenstein varieties, so similar connectedness results can also be proved if S/I_X quasi-Gorenstein.

To give an idea, the Stanley-Reisner ring of an orientable manifold Δ is quasi-Gorenstein, while it is Gorenstein $\Leftrightarrow$ Δ is a sphere.

Going back to the 27 lines $C \subseteq \mathbb{P}^3$ on a smooth cubic, we had that $C \subseteq \mathbb{P}^3$ was a complete intersection line arrangement of type $(3, 9)$: as predicted by our results, we already noticed that $G(C)$ is 10-connected $(10 = 3 + 9 - 2)$. In this case $G(C)$ is also 10-regular, and therefore not 11-connected, but this is not true for any complete intersection line arrangement:

Consider f and g homogeneous polynomials of degrees d and e in $\Bbbk[X_1, X_2, X_3]$. If f and g are general enough, they will form a complete intersection consisting in de distinct points in $\mathbb{P}^2$. Their cone will be a complete intersection line arrangement $C \subseteq \mathbb{P}^3$ of type (d, e) consisting of de lines passing through a point. In this case $G(C)$ is the complete graph on de vertices, so it is not $(d + e - 2)$-regular and it is $(de - 1)$-connected.

Planar singularities

Definition

A projective curve $C \subseteq \mathbb{P}^n$ has a planar singularity at a point $P \in C$ if $\dim_{\Bbbk} T_P C \leq 2$.

Remark

If a projective curve lies on a smooth surface, it has only planar singularities.

Theorem (Benedetti, Di Marca, _)

Let $C \subseteq \mathbb{P}^3$ be a complete intersection line arrangement of type (d, e). If C has only planar singularities, then the dual graph $G(C)$ is $(d + e - 2)$-regular. In particular, it is not $(d + e - 1)$-connected (while it is $(d + e - 2)$-connected by the previous results).

THANKS FOR YOUR ATTENTION!