Singularities and Hilbert functions

Matteo Varbaro (University of Genoa, Italy)
Joint with Hai Long Dao and Linquan Ma

Workshop on Commutative Algebra and Algebraic Geometry
in Prime Characteristic, ICTP Trieste 2023

Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be a standard graded algebra over a field $R_0 = K$. The *Hilbert Function* and *Hilbert Series* of R are

$$\mathrm{HF}_R(i) = \dim_K R_i \ \forall\ i \in \mathbb{N}, \quad \mathrm{HS}_R(t) = \sum_{i\in\mathbb{N}} \mathrm{HF}_R(i)t^i \in \mathbb{Z}[[t]].$$

If $d = \dim R$, Hilbert proved that

$$\mathrm{HS}_R(t) = \frac{h_R(t)}{(1-t)^d}$$

where $h_R(t) = h_0 + h_1 t + h_2 t^2 + \ldots + h_s t^s \in \mathbb{Z}[t]$ is the *h-polynomial* of R. $(h_0 = 1, h_1, h_2, \ldots, h_s)$ the *h-vector* of R.

Remark

While $\mathrm{HS}_R(t) \in \mathbb{N}[[t]]$, $h_R(t)$ may *not* belong to $\mathbb{N}[t]$. If $d = \dim R = 0$, then $h_R(t) = \mathrm{HS}(t) \in \mathbb{N}[t]$.

If $\ell \in R_1$ is an R-regular element, then $h_{R/(\ell)}(t) = h_R(t)$, so if R is Cohen-Macaulay we have $h_i \geq 0$ for all i. Without the CM assumption things change:

Example

Let $S = K[x_i, y_i : i = 1, \ldots, r+1]$ and $I \subset S$ the ideal

$$I = (x_1, \ldots, x_{r+1})^2 + (x_1 y_1 + \ldots + x_{r+1} y_{r+1}).$$

$R = S/I$ has dimension $(r+1)$, depth $R = r$ and has h-vector

$$(1, r+1, -1) \quad (h_2 < 0).$$

Such an R is even Buchsbaum (in particular R is generalized CM).

Remark

The integer $e(R) = h_R(1) = h_0 + h_1 + h_2 + \ldots$, being the *multiplicity of* R, is always positive.

Let $S = K[X_1, \ldots, X_n]$, where $n = \dim_K R_1$. Hence $R \cong S/I$ for some homogeneous ideal $I \subset S$ which *contains no linear forms*. Let c denote the height of I.

Remark

Since $I_1 = \{0\}$ we have:

- $h_1 = c$.
- $h_2 \geq 0 \iff \dim_K I_2 \leq \binom{c+1}{2}$.

If $I_j = \{0\}$ for all $j < k$, then $h_k \geq 0 \iff \dim_K I_k \leq \binom{c+k-1}{k}$.

Motivations

Theorem (Murai-Terai)

Let R be a Stanley-Reisner ring (i.e. I is generated by squarefree monomials of S) satisfying *Serre condition* (S_r). Then $h_i \geq 0$ for all $i \leq r$ and $e(R) \geq h_0 + h_1 + \ldots + h_{r-1}$. Furthermore, if $h_i = 0$ for some $i \leq r$, then R is Cohen-Macaulay.

For $r \in \mathbb{N}$, we recall that R *satisfies the Serre condition* (S_r) if:

$$\operatorname{depth} R_{\mathfrak{p}} \geq \min\{\dim R_{\mathfrak{p}}, r\} \quad \forall\, \mathfrak{p} \in \operatorname{Spec} R.$$

Remark

If R is generalized Cohen-Macaulay then R satisfies (S_r) if and only if depth $R \geq r$. Hence, by the previous example, (S_r) alone is not enough to infer $h_i \geq 0$ for all $i \leq r$, not even assuming Buchsbaum.

The condition MT_r

We say that $R = S/I$ *satisfies the condition* MT_r if

$$\operatorname{reg} \operatorname{Ext}_S^{n-i}(R, S(-n)) \leq i - r \quad \forall\, i = 0, \ldots, \dim R - 1.$$

This notion is good for several reasons:

- MT_r does not depend on S.
- MT_r is preserved by taking general hyperplane sections.
- MT_r is preserved by saturating.

The condition MT_r

Lemma (Murai-Terai, Dao-Ma-_)

If R satisfies MT_r, then

- $h_i \geq 0$ for all $i \leq r$.
- $h_r + \ldots + h_s \geq 0$, or equivalently $e(R) \geq h_0 + \ldots + h_{r-1}$.

Furthermore, if $\operatorname{reg} R < r$ or $h_i = 0$ for some $i \leq r$, then R is CM.

The main result

Theorem (Dao-Ma-_)

Assume that $R = S/I$ has dimension d and satisfies Serre condition (S_r), and suppose either

- K has characteristic 0 and R is Du Bois in codimension $d-1$ (e.g. (R_{d-1})), or
- K has characteristic 0, R is Du Bois in codimension $d-2$ (e.g. (R_{d-2})) and $d = r+1$, or
- K has positive characteristic and R is F-pure.

Then R satisfies condition MT_r. In particular, $h_i \geq 0$ for all $i \leq r$ and $e(R) \geq h_0 + h_1 + \ldots + h_{r-1}$. Furthermore, if $\operatorname{reg} R < r$, or if $h_i = 0$ for some $i \leq r$, then R is Cohen-Macaulay.

Problems

Stanley-Reisner rings are Du Bois in characteristic 0 and F-pure in positive characteristic, so we recover Murai-Terai result. However by results of Kummini-Murai MT_r holds whenever I is a monomial ideal and S/I satisfies (S_r) condition.

Questions:

Is it true that R satisfies MT_r if R satisfies (S_r) and either

1. K has characteristic 0 and R_{red} is Du Bois in codimension $r-1$,
2. K has characteristic 0 and R is Du Bois in codimension $r-2$, (in particular, if R is normal and satisfies (S_3), is $h_3 \geq 0$?),
3. K has positive characteristic and R_{red} is F-pure, or
4. K has positive characteristic and R is F-injective???

A new F-singularity

Suppose that K has positive characteristic:

Definition

We say that R is *deformation equivalent to an F-pure ring (deF-pure)* if there exist a domain A, essentially of finite type over K, and a flat finitely generated A-algebra R_A such that all fibres are standard graded, one fibre is R and one is F-pure. In other words, there exist prime ideals $\mathfrak{p}, \mathfrak{q} \in A$ such that $R_A \otimes_A \kappa(\mathfrak{p}) \cong R$ and $R_A \otimes_A \kappa(\mathfrak{q})$ is F-pure.

Remark

If R is F-pure, then it is de$F$-pure ($A = K$, $R_A = R$, $\mathfrak{p} = \mathfrak{q} = \{0\}$).

Proposition (Dao, Ma, _)

If R is de$F$-pure and satisfies (S_r), then it satisfies MT_r.

A new F-singularity

Recall that $R = S/I$ where $S = K[X_1, \ldots, X_n]$ and $I \subset S$ is a homogeneous ideal. Suppose that $\mathrm{in}(I)$ is a squarefree monomial ideal for some term order. One can check that R is de$F$-pure.

Example

If $n = 3$ and $I = (X_1^3 + X_2^3 + X_3^3 - X_1X_2X_3) \subset S$, then $R = S/I$ is de$F$-pure: taking $A = K[t]$, $J = (tX_1^3 + tX_2^3 + tX_3^3 - X_1X_2X_3)$ and $R_A = S[t]/J$, one has $R_A \otimes_A \kappa((t-1)) \cong R_A/(t-1) \cong S/I = R$. $R_A \otimes_A \kappa((t)) \cong R_A/(t) \cong S/(X_1X_2X_3)$ (that is F-pure). Note that, if $\mathrm{char}(K) = 5$, R is not even F-injective.

Question

Is an F-injective standard graded K-algebra de$F$-pure?

THANK YOU !