Gröbner deformations

Matteo Varbaro (University of Genoa, Italy)

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Notation and basic definitions

- $S = \mathbb{C}[X_0, \ldots, X_n]$, $\mathbb{P}^n$ n-dimensional projective space over $\mathbb{C}$.
- For any subset $F \subset S$ of homogeneous polynomials,

$$\mathcal{Z}(F) = \{[P] \in \mathbb{P}^n : f(P) = 0 \ \forall \ f \in F\} \subset \mathbb{P}^n.$$

- For any subset $X \subset \mathbb{P}^n$,

$$\mathcal{I}(X) = \{f \in S : f(P) = 0 \ \forall \ [P] \in X\} \subset S.$$

Subsets of $\mathbb{P}^n$ like $\mathcal{Z}(F)$ are called *algebraic sets*. As it turns out, $\mathcal{I}(X)$ is a homogeneous ideal of S.

Notice that, by Hilbert bases theorem, $I = (F)$ is generated by a finite number of homogeneous polynomials $f_1, \ldots, f_m \in S$, so:

$$\mathcal{Z}(I) = \mathcal{Z}(F) = \mathcal{Z}(\{f_1, \ldots, f_m\})$$

Notation and basic definitions

The previous definitions do not supply a 1-1 correspondence between algebraic subsets of $\mathbb{P}^n$ and homogeneous ideals of S:

Example

If $I = (X_0 - X_1), \ J = (X_0^2 - 2X_0X_1 + X_1^2) \subset \mathbb{C}[X_0, X_1, X_2]$, then $I \neq J$ but both $\mathcal{Z}(I)$ and $\mathcal{Z}(J)$ are the line

$$\{[s; s; t] \in \mathbb{P}^2 : (s, t) \in \mathbb{C}^2 \setminus \{(0, 0)\}.$$

The Nullstellensatz of Hilbert and the fundamental theorem of algebra say that $\mathcal{Z}(\cdot)$ and $\mathcal{I}(\cdot)$ give a 1-1 correspondence between:

homogeneous radical proper ideals of S and algebraic subsets of $\mathbb{P}^n$.

Monomial orders

The study of homogeneous ideals of $S = \mathbb{C}[X_0, \ldots, X_n]$, so, is strictly related to the study of the algebraic subsets of $\mathbb{P}^n$.

A useful method to study the ideals of S comes from Gröbner bases theory: to describe it, let Mon(S) be the set of monomials of S:

$$\text{Mon}(S) = \{X_0^{u_0} X_1^{u_1} \cdots X_n^{u_n} : (u_0, u_1, \ldots, u_n) \in \mathbb{N}^{n+1}\}.$$

Definition

A *monomial order* on S is a total order $<$ on Mon(S) such that:

(i) $1 \leq \mu$ for every $\mu \in \text{Mon}(S)$;

(ii) If $\mu_1, \mu_2, \nu \in \text{Mon}(S)$ such that $\mu_1 \leq \mu_2$, then $\mu_1 \nu \leq \mu_2 \nu$.

Notice that, if $<$ is a monomial order on S and μ, ν are monomials such that $\mu | \nu$, then $\mu \leq \nu$: indeed $1 \leq \nu/\mu$, so

$$\mu = 1 \cdot \mu \leq (\nu/\mu) \cdot \mu = \nu.$$

Monomial orders

Typical examples of monomial orders are the following: given monomials $\mu = X_0^{u_0} \cdots X_n^{u_n}$ and $\nu = X_0^{v_0} \cdots X_n^{v_n}$ we define:

- The *lexicographic order* (Lex) by $\mu <_{\text{Lex}} \nu$ iff $u_k < v_k$ for some k and $u_i = v_i$ for any $i < k$.
- The *degree lexicographic order* (DegLex) by $\mu <_{\text{DegLex}} \nu$ iff $\deg(\mu) < \deg(\nu)$ or $\deg(\mu) = \deg(\nu)$ and $\mu <_{Lex} \nu$.
- The *(degree) reverse lexicographic order* (RevLex) by $\mu <_{\text{RevLex}} \nu$ iff $\deg(\mu) < \deg(\nu)$ or $\deg(\mu) = \deg(\nu)$ and $u_k > v_k$ for some k and $u_i = v_i$ for any $i > k$.

Example

In $\mathbb{C}[X_0, X_1, X_2]$ we have $X_0X_2 >_{\text{Lex}} X_1^2$, while $X_1^2 >_{\text{RevLex}} X_0X_2$.

Monomial orders

Remark

The fact that a monomial order refines the divisibility partial order on $\mathrm{Mon}(S)$, together with the Hilbert basis theorem, makes a monomial order on S a well total order on $\mathrm{Mon}(S)$. This is the starting point for the theory of Gröbner bases.

Note that, fixed a monomial order $<$ on S, every nonzero polynomial $f \in S$ can be written uniquely as

$$f = a_1\mu_1 + \ldots + a_k\mu_k$$

with $a_i \in \mathbb{C} \setminus \{0\}$, $\mu_i \in \mathrm{Mon}(S)$ and $\mu_1 > \mu_2 > \ldots > \mu_k$.

Definition

The *initial monomial* of f is $\mathrm{in}(f) = \mu_1$.

Monomial orders

Example

If $f = X_0 + 2X_1X_3 - 3X_2^2$, we have:

- $\mathrm{in}(f) = X_0$ with respect to Lex.
- $\mathrm{in}(f) = X_1X_3$ with respect to DegLex.
- $\mathrm{in}(f) = X_2^2$ with respect to RevLex.

Example

If $f = X^2 + XY + Y^2 \in \mathbb{C}[X, Y]$, we have:

- $\mathrm{in}(f) = X^2$ if $X > Y$.
- $\mathrm{in}(f) = Y^2$ if $Y > X$.

In particular, $XY \neq \mathrm{in}(f)$ for all monomial orders.

Gröbner bases

Definition

If I is an ideal of $S = \mathbb{C}[X_0, \ldots, X_n]$, then the monomial ideal $\mathrm{in}(I) = (\{\mathrm{in}(f) : f \in I\}) \subset S$ is called the *initial ideal* of I.

Definition

Polynomials $f_1, \ldots, f_m$ of an ideal $I \subset S$ are a *Gröbner basis* of I if $\mathrm{in}(I) = (\mathrm{in}(f_1), \ldots, \mathrm{in}(f_m))$.

Example

Consider the ideal $I = (f_1 = X_0^2 - X_1^2, f_2 = X_0X_2 - X_1^2)$ of $\mathbb{C}[X_0, X_1, X_2]$. For Lex the polynomials f_1, f_2 are not a Gröbner basis of I, indeed $X_0X_1^2 = \mathrm{in}(X_2f_1 - X_0f_2)$ is a monomial of $\mathrm{in}(I)$ which is not in the ideal $(\mathrm{in}(f_1) = X_0^2, \mathrm{in}(f_2) = X_0X_2)$. For RevLex, it turns out that $\mathrm{in}(I) = (\mathrm{in}(f_1) = X_0^2, \mathrm{in}(f_2) = X_1^2)$, so f_1 and f_2 are a Gröbner basis of I in this case.

Remark

As we saw, a system of generators of I may not be a Gröbner basis of I. As it turns out, instead, a Gröbner basis of I is always a system of generators of I. Once again, the Hilbert basis theorem implies that, at least, any ideal of S has a finite Gröbner basis.

There is a way to compute a Gröbner basis of an ideal I (and so $\mathrm{in}(I)$) starting from a system of generators of I, namely the *Buchsberger's algorithm*. Since monomial ideals are much simpler to study than polynomial ideals, it is important to relate properties of the ideal I with properties of the monomial ideal $\mathrm{in}(I)$.

For such goals it is useful to see the passage from I to $\mathrm{in}(I)$ as a deformation. This is a bit technical, so we skip the interpretation and see what has been proved so far thanks to it.

Immediate remarks

For any monomial order on S and ideal $I \subset S$ it is easy to check:

- If $\text{in}(I)$ is a prime ideal, then I is a prime ideal.
- If $\text{in}(I)$ is a radical ideal, then I is a radical ideal.

The viceversa of the statements above do not hold: in fact there is plenty of prime ideals I such that $\text{in}(I)$ is not even radical.

From now on, we fix a monomial order $<$ on S, a homogeneous ideal $I \subset S$, and set $X = \mathcal{Z}(I)$ and $\text{in}(X) = \mathcal{Z}(\text{in}(I))$.

Let us recall than an algebraic set is *irreducible* if it is not the union of two proper algebraic subsets. It turns out that every algebraic set can be written in a unique way as a finite union of irreducible algebraic sets (its *irreducible components*). Also, an irreducible algebraic set is connected.

Connectedness properties

Although for a radical homogeneous ideal being prime is equivalent to its zero locus being irreducible, it is not true that

$$\mathrm{in}(X) \text{ irreducible} \implies X \text{ irreducible}$$

(think at $I = ((X_0 + X_1)(X_0 - X_1)) \subset \mathbb{C}[X_0, X_1]$ and Lex).

Of course "X irreducible $\implies$ in(X) irreducible" fails as well (think at $I = (X_0X_1X_2 + X_1^3 + X_2^3) \subset \mathbb{C}[X_0, X_1, X_2]$ and Lex).

What is true is the following:

Theorem (_, 2007)

If X is connected, then in(X) is connected. Furthermore, the converse holds if in(I) is radical.

Connectedness properties

For another similar result on algebraic sets, let us recall that the *dual graph* of an algebraic set Y is the simple graph having:

- As vertices, $\{1, \dots, s\}$ where $Y_1, \dots, Y_s$ are the irreducible components of Y.
- As edges, $\{i, j\}$ if $\operatorname{codim}_Y(Y_i \cap Y_j) = 1$.

Theorem (_, 2007, _-Nadi, 2019)

If the dual graph of X is connected, then the dual graph of $\mathrm{in}(X)$ is connected. Furthermore, the converse holds if $\mathrm{in}(I)$ is radical.

For the last part of this talk, we need to recall the notion of *scheme*: to a homogeneous ideal $J \subset S = \mathbb{C}[X_0, \dots, X_n]$, we associate a pair $(Y, \mathcal{O}_Y)$, where $Y = \mathcal{Z}(J) \subset \mathbb{P}^n$ in such a way that also the non-reduced part of J is taken in account:

Here $\mathcal{O}_Y$ is a sheaf of rings, defined so that for $i = 0, \dots, n$

$$\mathcal{O}_Y(\mathcal{D}_Y(X_i)) \cong \{f/X_i^m \ : \ m \in \mathbb{N},\ f \in S/J,\ \deg(f) = m\},$$

where $\mathcal{D}_Y(X_i) = \{[P] = [P_0, \dots, P_n] \in Y : P_i \neq 0\}$.

As it turns out, there is a 1-1 correspondence between homogeneous *saturated* ideals of S and *projective subschemes* of $\mathbb{P}^n$ (that we will call also *varieties*).

Gröbner degenerations in general

An important role is played by the twisted sheaves $\mathcal{O}_Y(k)$ where k varies among the integers. While $\mathcal{O}_Y(0) \cong \mathcal{O}_Y$ is an intrinsic invariant of the scheme $(Y, \mathcal{O}_Y)$, for $k \neq 0$ $\mathcal{O}_Y(k)$ depends on the embedding $Y \subset \mathbb{P}^n$.

Therefore the twisted sheaves are important to study the geometry of embedded projective schemes. In particular, from the numbers

$$h_k^i(Y) = \dim_{\mathbb{C}} H^i(Y, \mathcal{O}_Y(k)),$$

for $i, k \in \mathbb{Z}$, can be read off relevant invariants such as the Castelnuovo-Mumford regularity of $Y \subset \mathbb{P}^n$, the arithmetically Cohen-Macaulayness of $Y \subset \mathbb{P}^n$, but also intrinsic properties such as the Cohen-Macaulayness of Y.

Gröbner degenerations in general

Since the 80's it has been know that $h^i_k(X) \leq h^i_k(\text{in}(X))$ for all $i, k \in \mathbb{Z}$. In particular, if $\text{in}(X) \subset \mathbb{P}^n$ is arithmetically CM, then $X \subset \mathbb{P}^n$ is arithmetically CM as well. This is a useful way to prove that schemes like Grasmannians or determinantal varieties are arithmetically CM with respect to their natural embeddings.

This circle of ideas was initiated by De Concini, Eisenbud and Procesi in the 80's and examples where the above implication cannot be reversed have soon be found. However, many natural varieties like the above have an extra property: *their initial scheme is reduced*. Thus it started to circulate the question whether the equivalence

$$\text{in}(X) \subset \mathbb{P}^n \text{ is arithmetically CM iff } X \subset \mathbb{P}^n \text{ is arithmetically CM}$$

holds under the assumption that $\text{in}(X)$ is reduced...

Sometimes degenerative processes are not so bad!

Later on Herzog stated explicitly a stronger conjecture:

Conjecture (Herzog)

If $\text{in}(X)$ is reduced, then X and $\text{in}(X)$ have the same extremal Betti numbers.

The extremal Betti numbers of a scheme $Y \subset \mathbb{P}^n$ are special values of the $h^i_k(Y)$, which besides the arithmetically CMness establish also the Castelnuovo-Mumford regularity. The above formulation has been probably suggested by results of Bayer and Stillman of 1987 and Bayer, Charalambous and Popescu of 1998, who proved that X and $\text{in}(X)$ have the same extremal Betti numbers provided the coordinates are generic (w.r.t. X) and the monomial order is RevLex.

Theorem (Conca-_, 2018)

If $\text{in}(X)$ is reduced, then $h^i_k(X) = h^i_k(\text{in}(X))$ for all $i, k \in \mathbb{Z}$.

Sometimes degenerative processes are not so bad!

The above result suggests that varieties having a radical initial ideal should have particularly nice properties: as an example, we recently proposed the following:

Conjecture (Constantinescu, De Negri, _, 2019)

If $X \subset \mathbb{P}^n$ is a smooth projective variety such that $H^i(X, \mathcal{O}_X) \neq 0$ for some $i \in \mathbb{N}$, then $\mathrm{in}(X)$ is not reduced.

We proved it in a number of cases, for example if $<$ is RevLex or if X is an elliptic curve defined over a real number field, however for the moment the conjecture is open even for Calabi-Yau's...

THANKS FOR YOUR ATTENTION!