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Abstract

Let $\mathfrak{a}$ be a homogeneous ideal of a polynomial ring R in n variables over a field $\Bbbk$. Assume that $\operatorname{depth}(R/\mathfrak{a}) \geqslant t$, where t is some number in $\{0, \ldots, n\}$. A result of Peskine and Szpiro says that if $\operatorname{char}(\Bbbk) > 0$, then the local cohomology modules $H^i_{\mathfrak{a}}(M)$ vanish for all $i > n - t$ and all R-modules M. In characteristic 0, there are counterexamples to this for all $t \geqslant 4$. On the other hand, when $t \leqslant 2$, by exploiting classical results of Grothendieck, Lichtenbaum, Hartshorne and Ogus it is not difficult to extend the result to any characteristic. In this paper we settle the remaining case; specifically, we show that if $\operatorname{depth}(R/\mathfrak{a}) \geqslant 3$, then the local cohomology modules $H^i_{\mathfrak{a}}(M)$ vanish for all $i > n - 3$ and all R-modules M, whatever the characteristic of $\Bbbk$ is.

1. Introduction

In his seminar on local cohomology [Gro67, p. 79], Grothendieck posed the problem of finding conditions under which, for a fixed positive integer c, the local cohomology modules $H^i_{\mathfrak{a}}(R)$ vanish for every $i > c$, where $\mathfrak{a}$ is an ideal in a ring R. In other words, one seeks conditions under which the cohomological dimension $\operatorname{cd}(R, \mathfrak{a}) \leqslant c$. Since then, many mathematicians have worked on this problem (for instance, see [Har68, Ogu73, HS77, Fal80, HL90, Lyu06]). In the same spirit, we will study the relationships between cohomological and projective dimensions. Before explaining the results of the paper, let us summarize some essential known facts about this subject.

The two earliest results are both due to Grothendieck (see [Gro67]): they are essential, as they fix the range in which we must look for the natural number $\operatorname{cd}(R, \mathfrak{a})$:

$$\operatorname{ht}(\mathfrak{a}) \leqslant \operatorname{cd}(R, \mathfrak{a}) \leqslant \dim R.$$

Afterwards, first Lichtenbaum and then Hartshorne [Har68], in more generality, settled the problem of characterizing when $\operatorname{cd}(R, \mathfrak{a}) \leqslant \dim R - 1$. Roughly speaking, they showed that a necessary and sufficient condition for this to happen is that $\dim R/\mathfrak{a} > 0$. Then, the next step should have been to describe when $\operatorname{cd}(R, \mathfrak{a}) \leqslant \dim R - 2$. In general, this case is still not understood. However, if R is a complete regular local ring containing a field, a necessary and sufficient condition is that the punctured spectrum of $R/\mathfrak{a}$ is connected. This has been shown in [HS77] in positive characteristic and in [Ogu73] in characteristic 0. In [HL90], a characteristic-free proof is given. Actually, if the ambient ring R is regular, $\operatorname{cd}(R, \mathfrak{a})$ can always be characterized in terms of the ring $R/\mathfrak{a}$ (see [HS77, Lyu06, Ogu73]). However, in all of these papers, the conditions described are quite difficult to verify.

A classical result of Peskine and Szpiro, given in [PS73], says that if $\mathfrak{a}$ is a perfect ideal of a regular local ring of characteristic $p > 0$, then $\operatorname{cd}(R, \mathfrak{a}) = \operatorname{ht}(\mathfrak{a})$. The proof is based on the

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flatness of the Frobenius map $R \to R$, which, by the work of Kunz [Kun69], is equivalent to R being regular. We observe that Peskine and Szpiro's idea works for all Noetherian rings of positive characteristic (see Corollary 2.2), exploiting the acyclicity criterion that Buchsbaum and Eisenbud obtained in [BE73] instead of the result of Kunz. (Indeed, this idea also works in some situations in characteristic 0; see Lemma 2.1.)

In characteristic 0 the situation is completely different. There are several instances of perfect ideals with high cohomological dimension; see Example 2.6. Such examples appear even in a regular ambient; thus we stick to the situation in which R is an n-dimensional regular local ring containing a field. In this case, if $\mathfrak{a}$ is a perfect ideal of height $n-2$, then $\operatorname{cd}(R, \mathfrak{a}) = n-2$. More generally, if $\operatorname{depth}(R/\mathfrak{a}) \geqslant 2$, then $\operatorname{cd}(R, \mathfrak{a}) \leqslant n-2$; see Proposition 3.1. We know examples of perfect ideals of height $n-4$ and cohomological dimension $n-3$, but we do not know any example of an ideal $\mathfrak{a} \subseteq R$ of projective dimension less than or equal to $n-3$ such that $\operatorname{cd}(R, \mathfrak{a}) > n-3$. Therefore we consider the case where $\operatorname{depth}(R/\mathfrak{a}) \geqslant 3$. We show in Proposition 3.2 that $H^i_{\mathfrak{a}}(R) = 0$ for all $i \geqslant n-1$ and $H^{n-2}_{\mathfrak{a}}(R)_{\mathfrak{p}} = 0$ for any prime ideal of R different from the maximal one. As a consequence, we get that the Lyubeznik numbers of a local ring A (defined in [Lyu93]), $\lambda_{i,j}(A)$, vanish for all $0 \leqslant j < \operatorname{depth}(A)$ and $i \geqslant j-1$; see Corollary 3.3. At this point, we focus on the special case where R is a polynomial ring in n variables over a field of characteristic 0 and $\mathfrak{a}$ is a homogeneous ideal such that $\operatorname{depth}(R/\mathfrak{a}) \geqslant 3$. The main result of the paper is that, under the above assumptions, $\operatorname{cd}(R, \mathfrak{a}) \leqslant n-3$ (see Theorem 3.5). In particular, if $\mathfrak{a}$ is perfect of height $n-3$, then $\operatorname{cd}(R, \mathfrak{a}) = n-3$. As a consequence, in Remark 3.7 we are able to give several examples of prime ideals $\mathfrak{a} \subseteq R$ such that $R/\mathfrak{a}$ is not set-theoretically Cohen–Macaulay, thereby generalizing a result of Singh and Walther in [SW05]. More such examples can be produced using Proposition 3.8.

2. Ideals with small cohomological dimension

Given an ideal $\mathfrak{a}$ of a Noetherian ring R, its *cohomological dimension*, denoted by $\operatorname{cd}(R, \mathfrak{a})$, is the smallest natural number c such that the local cohomology modules $H^i_{\mathfrak{a}}(M)$ vanish for all R-modules M and all $i > c$. As is well known, it suffices to check the condition for $M = R$. We have

$$\operatorname{cd}(R, \mathfrak{a}) \geqslant \operatorname{ht}(\mathfrak{a}),$$

and in this section we will find some ideals for which the above inequality is an equality. It is convenient to recall a definition: an ideal $\mathfrak{a} \subseteq R$ is *perfect* if the maximal length of an R-regular sequence in $\mathfrak{a}$, namely $\operatorname{grade}(\mathfrak{a})$, is equal to the projective dimension of $R/\mathfrak{a}$ (in particular, a perfect ideal has finite projective dimension). If R is a regular local ring, then $\mathfrak{a}$ is perfect if and only if $R/\mathfrak{a}$ is Cohen–Macaulay. The following fact has many interesting corollaries.

Lemma 2.1. *Let R be a Noetherian ring and $\mathfrak{a} \subseteq R$ a perfect ideal. Assume that for all integers $k \in \mathbb{N}$ there is a ring-homomorphism $\phi_k : R \to R$ such that:*

(1) $\phi_0 = 1_R$;

(2) $\phi_i(\mathfrak{a}) \subseteq \phi_j(\mathfrak{a})$ *whenever* $i \geqslant j$;

(3) *the inverse system of ideals* $\{\phi_k(\mathfrak{a})R\}_{k\in\mathbb{N}}$ *is cofinal with* $\{\mathfrak{a}^k\}_{k\in\mathbb{N}}$.

Then $\operatorname{cd}(R, \mathfrak{a}) = \operatorname{ht}(\mathfrak{a})$.

Proof. Set $\operatorname{grade}(\mathfrak{a}) = g$. Since for every i there exists j such that $\mathfrak{a}^j \subseteq \phi_i(\mathfrak{a})R \subseteq \mathfrak{a}$, we have $\operatorname{grade}(\phi_k(\mathfrak{a})R) = g$ for all $k \in \mathbb{N}$. So $\phi_k(\mathfrak{a})R$ is a perfect ideal for all $k \in \mathbb{N}$ by [BV80, Theorem 3.5].

Therefore

$$\mathrm{Ext}^i_R(R/\phi_k(\mathfrak{a})R, R) = 0 \quad \text{for all } i > g,\ k \in \mathbb{N}.$$

We infer that $\mathrm{cd}(R, \mathfrak{a}) \leqslant g$ by the identity

$$H^i_{\mathfrak{a}}(R) \cong \varinjlim \mathrm{Ext}^i(R/\phi_k(\mathfrak{a}), R).$$

The proof is then complete because $\mathrm{cd}(R, \mathfrak{a}) \geqslant \mathrm{ht}(\mathfrak{a}) \geqslant \mathrm{grade}(\mathfrak{a})$. □

The first consequence of Lemma 2.1 is the promised extension of [PS73, Proposition 4.1].

Corollary 2.2. *Let R be a Noetherian ring of positive characteristic. If $\mathfrak{a} \subseteq R$ is a perfect ideal, then* $\mathrm{cd}(R, \mathfrak{a}) = \mathrm{ht}(\mathfrak{a})$.

Proof. This follows from Lemma 2.1, by considering the kth iteration of the Frobenius map as ϕ_k. □

Of course, the converse of the above corollary does not hold, since the cohomological dimension of $\mathfrak{a}$ is an invariant of the radical of $\mathfrak{a}$.

Example 2.3. Notice that Corollary 2.2 does not hold if we just assume that $R/\mathfrak{a}$ is Cohen–Macaulay: for instance, let $R = \mathbb{k}[x, y]/(xy)$ and $\mathfrak{a} = (x) \subseteq R$. Note that $R/\mathfrak{a} \cong \mathbb{k}[y]$ is Cohen–Macaulay, $\mathrm{ht}(\mathfrak{a}) = 0$, and

$$\cdots \xrightarrow{\cdot y} R \xrightarrow{\cdot x} R \xrightarrow{\cdot y} R \xrightarrow{\cdot x} \mathfrak{a} \to 0$$

is an infinite minimal free resolution of $\mathfrak{a}$. Considering the homomorphism $R \to \mathbb{k}[x]$ mapping x to itself and y to zero and viewing $M = \mathbb{k}[x]$ as an R-module, one has $H^1_{\mathfrak{a}}(M) \cong H^1_{(x)\mathbb{k}[x]}(\mathbb{k}[x]) \neq 0$. In particular, $\mathrm{cd}(R, \mathfrak{a}) = 1 > 0 = \mathrm{ht}(\mathfrak{a})$.

We note two further consequences of Lemma 2.1.

Corollary 2.4. *Let A be a Noetherian ring and Γ a finitely generated commutative monoid. If $\mathfrak{a} \subseteq R = A[\Gamma]$ is a perfect ideal generated by a subset of Γ, then* $\mathrm{cd}(R, \mathfrak{a}) = \mathrm{ht}(\mathfrak{a})$.

Proof. Once again, we want to use Lemma 2.1. For this purpose, we look at ϕ_k induced by the monoid homomorphisms $\Gamma \xrightarrow{\cdot k} \Gamma$, which satisfy the assumptions of Lemma 2.1. □

Corollary 2.5. *Let R be a Noetherian ring of positive characteristic and (r_{ij}) an $m \times n$ matrix with entries in R. Let $\mathfrak{a}$ be the ideal generated by the* $\min\{m, n\}$*-minors of (r_{ij}). If* $\mathrm{ht}(\mathfrak{a}) = |n - m| + 1$, *then* $\mathrm{cd}(\mathfrak{a}) = \mathrm{ht}(\mathfrak{a})$.

Proof. Such an ideal $\mathfrak{a}$ is resolved by the Eagon–Northcott complex, so it is perfect. Therefore the conclusion follows from Corollary 2.2. □

Corollary 2.5 (and therefore also Corollary 2.2) does not hold in characteristic 0, as shown by the following example.

Example 2.6. Let $\mathbb{k}$ be a field of characteristic 0, (x_{ij}) an $m \times n$ matrix of indeterminates over $\mathbb{k}$, and $R = \mathbb{k}[x_{ij}]$. If $\mathfrak{a} \subseteq R$ is the ideal generated by the t-minors of (x_{ij}), for $t \leqslant \min\{m, n\}$, then $\mathfrak{a}$ is a perfect ideal of $\mathrm{ht}(\mathfrak{a}) = (m - t + 1)(n - t + 1)$. However, by [BS90], we have $\mathrm{cd}(R, \mathfrak{a}) = mn - t^2 + 1$. Therefore

$$\mathrm{cd}(R, \mathfrak{a}) - \mathrm{ht}(\mathfrak{a}) = (m + n - 2t)(t - 1),$$

which, unless $m = n = t$ or $t = 1$, is a positive integer.

3. Characteristic 0

Notice that in Example 2.6 we have $\dim R/\mathfrak{a} = (t-1)(m+n-t+1)$. As one can check, $d = \dim R/\mathfrak{a}$ can be any natural number different from 1 and 2. Furthermore, if $d \in \{0, 3\}$, then we are forced to be in the special cases where $\operatorname{cd}(R, \mathfrak{a}) = \operatorname{ht}(\mathfrak{a})$. However, if $d \geqslant 4$, numbers for which $\operatorname{cd}(R, \mathfrak{a}) > \operatorname{ht}(\mathfrak{a})$ can always be chosen (for example, $t = m = 2$ and $n = d-1$, the case where $\mathfrak{a}$ defines the Segre product $\mathbb{P}^1 \times \mathbb{P}^{d-2}$ inside $\mathbb{P}^{2d-3}$). In this section, we try to understand what happens in the remaining cases. Slightly more generally, we wonder whether we can deduce $\operatorname{cd}(R, \mathfrak{a}) \leqslant n - t$, knowing that $\operatorname{depth}(R/\mathfrak{a}) \geqslant t$ where $t \leqslant 3$. If $t = 0$, then the vanishing theorem of Grothendieck implies $\operatorname{cd}(R/\mathfrak{a}) \leqslant n$. If $t = 1$, then one can show that $\operatorname{cd}(R, \mathfrak{a}) \leqslant n-1$ using the theorem of Hartshorne and Lichtembaum [Har68, Theorem 3.1]. When $t = 2$, one can show that $\operatorname{cd}(R, \mathfrak{a}) \leqslant n-2$ provided that R is a regular local ring containing a field, by exploiting a result of Ogus [Ogu73, Corollary 2.11] and one of Hartshorne [Har62, Proposition 2.1]. We will present the proof of this last case.

Proposition 3.1. *Let $(R, \mathfrak{m})$ be an n-dimensional regular local ring containing a field, and let $\mathfrak{a} \subseteq R$ be an ideal. If $\operatorname{depth}(R/\mathfrak{a}) \geqslant 2$, then $\operatorname{cd}(R, \mathfrak{a}) \leqslant n-2$.*

Proof. Suppose $\Bbbk = R/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of R. If $\operatorname{char}(\Bbbk) > 0$, we already know the result, so we can assume that $\operatorname{char}(\Bbbk) = 0$. Take a faithfully flat homomorphism from $(R, \mathfrak{m})$ to a regular local ring $(S, \mathfrak{n})$ such that $S/\mathfrak{n}$ is the algebraic closure of $\Bbbk$ (such a thing exists: it is a suitable *gonflement* of R; see [Bou08, ch. IX, Appendice 2]). Faithful flatness guarantees that S still contains a field, $\operatorname{depth}(R/\mathfrak{a}) = \operatorname{depth}(S/\mathfrak{a}S)$, and $\operatorname{cd}(R, \mathfrak{a}) = \operatorname{cd}(S, \mathfrak{a}S)$. Therefore we can assume that $\Bbbk$ is algebraically closed. Again, since $\widehat{R}$ is faithfully flat over R, we have $\operatorname{cd}(\widehat{R}, \mathfrak{a}\widehat{R}) = \operatorname{cd}(R, \mathfrak{a})$ and $\operatorname{depth}(\widehat{R}/\mathfrak{a}\widehat{R}) = \operatorname{depth}(R/\mathfrak{a})$. Thus it is harmless to assume that R is complete, so that $R \cong \Bbbk[[x_1, \ldots, x_n]]$ by the Cohen structure theorem. By [Har62, Proposition 2.1], $\operatorname{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}/\mathfrak{a}\}$ is connected. So [Ogu73, Corollary 2.11] yields the conclusion. □

It now remains to understand the case where $t = 3$: if $\mathfrak{a}$ is an ideal of a regular local ring such that $\operatorname{depth}(R/\mathfrak{a}) \geqslant 3$, is it true that $\operatorname{cd}(R, \mathfrak{a}) \leqslant n-3$?

Proposition 3.2. *Let $(R, \mathfrak{m})$ be an n-dimensional regular local ring containing a field, and let $\mathfrak{a} \subseteq R$ be an ideal. If $\operatorname{depth}(R/\mathfrak{a}) = k$, then $\dim(\operatorname{Supp}(H^{n-i}_{\mathfrak{a}}(R)) \leqslant i-2$ for all $0 \leqslant i < k$.*

Proof. Given $i < k$, we have to show that $(H^{n-i}_{\mathfrak{a}}(R))_{\mathfrak{p}} = H^{n-i}_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\operatorname{ht}(\mathfrak{p}) \leqslant n-i+1$. Let us denote by h the height of $\mathfrak{p}$. We can suppose that $h \geqslant n-i$, because otherwise $H^{n-i}_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ would automatically be 0 (since $\dim R_{\mathfrak{p}} = h < n-i$).

First, let us assume that $h = n-i$. Since $i < k = \operatorname{depth}(R/\mathfrak{a})$, $\mathfrak{p}$ is not a minimal prime of $R/\mathfrak{a}$. This implies that $\dim R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}} > 0$, which, using the Hartshorne–Lichtenbaum theorem, yields $H^{n-i}_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$.

So we can suppose $h = n-i+1$. A theorem of Ischebeck [Mat80, Theorem 17.1] yields

$$\operatorname{Ext}^0_R(R/\mathfrak{p}, R/\mathfrak{a}) = \operatorname{Ext}^1_R(R/\mathfrak{p}, R/\mathfrak{a}) = 0.$$

This means that $\operatorname{grade}(\mathfrak{p}, R/\mathfrak{a}) > 1$ and hence that $H^0_{\mathfrak{p}}(R/\mathfrak{a}) = H^1_{\mathfrak{p}}(R/\mathfrak{a}) = 0$. In particular,

$$H^0_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}) = H^1_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}) = 0,$$

that is, $\operatorname{depth}(R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}) \geqslant 2$. Since $R_{\mathfrak{p}}$ is an $(n-i+1)$-dimensional regular local ring, Proposition 3.1 yields

$$H^{n-i}_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0.$$

This concludes the proof. □

A first consequence of Proposition 3.2 concerns a fact about the Lyubeznik numbers of a local ring. We recall the definition as follows. Let A be a local ring which admits a surjection from an n-dimensional regular local ring containing a field. Let $\mathfrak{a}$ be the kernel of the surjection, and let $\Bbbk = R/\mathfrak{m}$. In [Lyu93], Lyubeznik proved that the Bass numbers $\lambda_{i,j}(A) = \dim_\Bbbk \operatorname{Ext}^i_R(\Bbbk, H^{n-j}_{\mathfrak{a}}(R))$ depend only on A, i and j, but neither on R nor on the surjection $R \to A$. For this reason, they are usually called the Lyubeznik numbers of A. Furthermore, they can be defined for any local ring containing a field, upon passing to the completion if needed. In [Lyu93] it was also shown that $\lambda_{i,j}(A) = 0$ whenever $j > d = \dim A$ or $i > j$ and that $\lambda_{d,d}(A) \neq 0$.

COROLLARY 3.3. *Let A be a local ring containing a field. If* $\operatorname{depth}(A) = k$, *then* $\lambda_{i-1,i}(A) = \lambda_{i,i}(A) = 0$ *for all* $0 \leqslant i < k$.

Proof. By [Lyu93, Corollary 3.6], the injective dimension of $H^i_{\mathfrak{a}}(R)$ is bounded above by the dimension of the support of $H^i_{\mathfrak{a}}(R)$, so the statement follows from Proposition 3.2. □

We do not know whether local rings as in Corollary 3.3 satisfy $\lambda_{i-2,i}(A) = 0$ for all $i < k$. Actually, this is related to the question we are investigating. However, we can provide an example for which $k > 3$ and $\lambda_{0,3}(A) \neq 0$.

Example 3.4. Let I be the homogeneous ideal of $S = \Bbbk[x_{pq} : p = 0, \dots, r; q = 0, \dots, s]$ defining the Segre product $\mathbb{P}^r \times \mathbb{P}^s \subseteq \mathbb{P}^{rs+r+s}$, where $\Bbbk$ is a field of characteristic 0 and $r > s \geqslant 1$. Let $\mathfrak{m}$ be the maximal irrelevant of S, $R = S_{\mathfrak{m}}$, $\mathfrak{a} = IR$ and $A = R/\mathfrak{a}$. We know from Example 2.6 that if $n = rs + r + s + 1$, then $H^{n-3}_{\mathfrak{a}}(R) \neq 0$. Moreover, if $\mathfrak{p} \in \operatorname{Spec} R$ is not maximal, then $H^{n-3}_{\mathfrak{a}}(R)_{\mathfrak{p}} = 0$; in fact, since $\mathbb{P}^r \times \mathbb{P}^s$ is smooth, $\mathfrak{a}R_{\mathfrak{p}} \subseteq R$ is generated by a regular sequence of length $n - r - s - 1 < n - 3$. Therefore $\dim(\operatorname{Supp}(H^{n-3}_{\mathfrak{a}}(R)) = 0$, so $H^{n-3}_{\mathfrak{a}}(R)$ is an injective R-module by [Lyu93]. Because it is supported at $\mathfrak{m}$, we have $H^{n-3}_{\mathfrak{a}}(R) \cong E^s$ for some $s > 0$, where E is the injective hull of $\Bbbk$ (as an R-module). Eventually, we have $\operatorname{Hom}_R(\Bbbk, E^s) \cong \Bbbk^s$, which implies $\lambda_{0,3}(A) \neq 0$.

Before stating the main result of the paper, let us introduce some notation. Let X be a projective scheme over a field $\Bbbk$ of characteristic 0. By $H^i_{\mathrm{DR}}(X)$ we mean algebraic de Rham cohomology, as defined in [Har75]. If $\Bbbk = \mathbb{C}$, we denote by X_h the analytic space associated to X. By [Har75, ch. IV, Theorem 1.1], we have $H^i(X_h, \mathbb{C}) \cong H^i_{\mathrm{DR}}(X)$, where $H^i(X_h, \mathbb{C})$ means singular cohomology with coefficients in $\mathbb{C}$.

THEOREM 3.5. *Let $\Bbbk$ be a field of any characteristic and $R = \Bbbk[x_1, \dots, x_n]$. If $\mathfrak{a} \subseteq R$ is a homogeneous ideal such that* $\operatorname{depth}(R/\mathfrak{a}) \geqslant 3$, *then* $\operatorname{cd}(R, \mathfrak{a}) \leqslant n - 3$. *In particular, if $R/\mathfrak{a}$ is a 3-dimensional Cohen–Macaulay ring, then* $\operatorname{cd}(R, \mathfrak{a}) = n - 3$.

Proof. Of course, we need to show this only when the characteristic of $\Bbbk$ is 0, since in positive characteristic it is true by the result of Peskine and Szpiro. The properties in the hypothesis and in the conclusion are preserved under flat extensions. So, since $\mathfrak{a}$ is finitely generated, we can assume that $\mathbb{Q} \subseteq \Bbbk \subseteq \mathbb{C}$ and, eventually, that $\Bbbk = \mathbb{C}$. That $\operatorname{cd}(R, \mathfrak{a}) \leqslant n - 3$ is equivalent to saying that $H^i(\mathbb{P}^{n-1} \setminus X, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on $X = \mathcal{V}_+(\mathfrak{a})$ and for all $i \geqslant n - 3$. This, by [Ogu73, Theorem 4.4], is equivalent to saying that $H^0_{\mathrm{DR}}(X) \cong H^0_{\mathrm{DR}}(\mathbb{P}^{n-1})$, $H^1_{\mathrm{DR}}(X) \cong H^1_{\mathrm{DR}}(\mathbb{P}^{n-1})$ and the de Rham depth of X is at least 2. The last condition is, in turn, equivalent to $\operatorname{Supp}(H^i_{\mathfrak{a}}(R)) \subseteq \{\mathfrak{m}\}$ for all $i \geqslant n - 2$, where $\mathfrak{m}$ is the maximal irrelevant ideal (see the proof of [Ogu73, Theorem 4.1]), and this is true by Proposition 3.2. The condition $H^0_{\mathrm{DR}}(X) \cong H^0_{\mathrm{DR}}(\mathbb{P}^{n-1})$ means that X is connected, which is the case because $\operatorname{depth}(R/\mathfrak{a}) \geqslant 3$; see [Har62, Proposition 2.1]. So we have to show that $H^1_{\mathrm{DR}}(X) \cong H^1_{\mathrm{DR}}(\mathbb{P}^{n-1}) = 0$. Since $H^1_{\mathrm{DR}}(X) \cong H^1(X_h, \mathbb{C})$, by the universal coefficient theorem it is enough to show that $H^1(X_h, \mathbb{Z})$ is zero. Let us consider the morphisms of sheaves $\mathbb{Z}_{X_h} \to \mathcal{O}_{X_h} \to (\mathcal{O}_{X_h})_{\mathrm{red}}$ (here $\mathbb{Z}_{X_h}$ denotes the locally constant sheaf on X_h associated to $\mathbb{Z}$). By the exponential sequence, we know that the composition

$$H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, (\mathcal{O}_{X_h})_{\mathrm{red}})$$

is injective, so $H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h})$ has to be injective too. (Actually, the exponential sequence makes sense also for non-reduced schemes; however, we prefer to follow a different route because of the lack of references.) By [Ser56], $H^1(X_h, \mathcal{O}_{X_h}) \cong H^1(X, \mathcal{O}_X)$. Furthermore, $H^1(X, \mathcal{O}_X) \cong H^2_{\mathfrak{m}}(R/\mathfrak{a})_0$, and the last vector space is zero because $\operatorname{depth}(R/\mathfrak{a}) \geqslant 3$. So $H^1(X_h, \mathbb{Z}) \cong H^1(X_h, \mathbb{Z}_{X_h})$ has to be zero. □

Remark 3.6. Under the assumption that X is smooth, where the situation is considerably simpler, we obtained the conclusion of Theorem 3.5 in [Var12]. Even in the smooth case, however, the above proof cannot be repeated to show that $\operatorname{depth}(R/\mathfrak{a}) \geqslant 4$ implies $\operatorname{cd}(R, \mathfrak{a}) \leqslant n-4$. The point is that

$$H^2(X_h, \mathbb{C}_{X_h}) \cong H^2(X_h, \mathcal{O}_{X_h}) \oplus H^1(X_h, \Omega^1_{X_h}) \oplus H^0(X_h, \Omega^2_{X_h});$$

so, owing to the presence of the middle term $H^1(X_h, \Omega^1_{X_h})$, $H^3_{\mathfrak{m}}(R/\mathfrak{a}) = 0$ does not imply $H^2(X_h, \mathbb{C}_{X_h}) = 0$.

Remark 3.7. In [SW05, Theorem 3.3], Singh and Walther used characteristic p methods to prove the following. If $E \subseteq \mathbb{P}^2$ is an elliptic curve defined over $\mathbb{Z}$, $\Bbbk$ is a field of characteristic 0 and $\mathfrak{a} \subseteq R = \Bbbk[x_0, \ldots, x_5]$ is the defining ideal of $E \times \mathbb{P}^1$, then $R/\mathfrak{b}$ is not Cohen–Macaulay for all homogeneous ideals $\mathfrak{b}$ with the same radical of $\mathfrak{a}$. Theorem 3.5 immediately yields a much more general fact: let C be a smooth curve of genus at least 1 and let Y be any projective scheme, both over a field $\Bbbk$ of characteristic 0. Let $\mathfrak{a} \subseteq R = \Bbbk[x_1, \ldots, x_N]$ be the defining ideal of $X = C \times Y \subseteq \mathbb{P}^{N-1}$. For all homogeneous $\mathfrak{b} \subseteq R$ such that $\sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}}$, $\operatorname{depth}(R/\mathfrak{b}) \leqslant 2$. To show this, we claim that $\operatorname{cd}(R, \mathfrak{a}) \geqslant N-2$, so that, since the cohomological dimension is independent of the radical, Theorem 3.5 would give the conclusion. For this purpose, we can assume $\Bbbk = \mathbb{C}$. GAGA and Hodge decomposition imply that $H^1(C_h, \mathbb{C}) \neq 0$; so $H^1(X_h, \mathbb{C}) \neq 0$ by the Kunneth formula, and thus $\operatorname{cd}(R, \mathfrak{a}) \geqslant N-2$ by a result of Hartshorne (see [Har70, Theorem 7.4, p. 148]).

The above remark shows how Theorem 3.5 can be used to produce ideals which are not set-theoretically Cohen–Macaulay. The following proposition gives further such examples.

Proposition 3.8. *Let $\Bbbk$ be a field of characteristic 0, $\mathfrak{a} \subseteq R = \Bbbk[x_1, \ldots, x_n]$ a graded ideal and $\mathfrak{m} = (x_1, \ldots, x_n)$ the maximal irrelevant. Assume that $\mathfrak{b} = \sqrt{\mathfrak{a}}$ is such that $X = \operatorname{Proj}(R/\mathfrak{b})$ is smooth. Then*

$$\dim_{\Bbbk} H^i_{\mathfrak{m}}(R/\mathfrak{a})_0 \geqslant \dim_{\Bbbk} H^i_{\mathfrak{m}}(R/\mathfrak{b})_0 \quad \textit{for all } i \geqslant 0.$$

Proof. For $i = 0$ this is trivial. By the exact sequences of graded R-modules

$$0 \to H^0_{\mathfrak{m}}(R/\mathfrak{a}) \to R/\mathfrak{a} \to \bigoplus_{i \in \mathbb{N}} H^0(X, \mathcal{O}_X(i)) \to H^1_{\mathfrak{m}}(R/\mathfrak{a}) \to 0,$$

$$0 \to H^0_{\mathfrak{m}}(R/\mathfrak{b}) \to R/\mathfrak{b} \to \bigoplus_{i \in \mathbb{N}} H^0(X, (\mathcal{O}_X)_{\mathrm{red}}(i)) \to H^1_{\mathfrak{m}}(R/\mathfrak{b}) \to 0,$$

the dimension of the $\Bbbk$-vector spaces $H^1_{\mathfrak{m}}(R/\mathfrak{a})_0$ and $H^1_{\mathfrak{m}}(R/\mathfrak{b})_0$ are, respectively, $\dim_{\Bbbk} H^0(X, \mathcal{O}_X) - 1$ and $\dim_{\Bbbk} H^0(X, (\mathcal{O}_X)_{\mathrm{red}}) - 1$, and obviously $\dim_{\Bbbk} H^0(X, \mathcal{O}_X) \geqslant \dim_{\Bbbk} H^0(X, (\mathcal{O}_X)_{\mathrm{red}})$. Note that so far we have not used the smoothness of X_{red}. For $i \geqslant 2$ we have to. As usual, it is harmless to assume $\Bbbk = \mathbb{C}$. Let us recall the isomorphisms of $\mathbb{C}$-vector spaces $H^i_{\mathfrak{m}}(R/\mathfrak{a})_0 \cong H^{i-1}(X, \mathcal{O}_X)$ and $H^i_{\mathfrak{m}}(R/\mathfrak{b})_0 \cong H^{i-1}(X, (\mathcal{O}_X)_{\mathrm{red}})$. Consider the natural maps of sheaves

$$\mathbb{C}_{X_h} \to \mathcal{O}_{X_h} \to (\mathcal{O}_{X_h})_{\mathrm{red}}.$$

These yield maps of $\mathbb{C}$-vector spaces

$$H^i(X_h, \mathbb{C}_{X_h}) \xrightarrow{\alpha} H^i(X_h, \mathcal{O}_{X_h}) \xrightarrow{\beta} H^i(X_h, (\mathcal{O}_{X_h})_{\mathrm{red}}).$$

The composition of these homomorphisms is surjective; indeed, by Hodge theory, $H^i(X_h, (\mathcal{O}_{X_h})_{\mathrm{red}})$ is the space of harmonic $(i, 0)$-forms, and Hodge decomposition tells us that $\beta\alpha$ maps a harmonic i-form to its $(i, 0)$ component (see the book of Arapura [Ara12] for terminology not explained here). Therefore $H^i(X_h, \mathcal{O}_{X_h}) \to H^i(X_h, (\mathcal{O}_{X_h})_{\mathrm{red}})$ is surjective. By [Ser56], there are isomorphisms of $\mathbb{C}$-vector spaces $H^i(X, \mathcal{O}_X) \cong H^i(X_h, \mathcal{O}_{X_h})$ and $H^i(X, (\mathcal{O}_X)_{\mathrm{red}}) \cong H^i(X_h, (\mathcal{O}_{X_h})_{\mathrm{red}})$, and thus we can conclude the proof. □

The smoothness assumption in Proposition 3.8 is necessary, as demonstrated by the following example due to Aldo Conca.

Example 3.9. Let $R = \mathbb{Q}[x_1, \ldots, x_6]$ and

$$\mathfrak{a} = (x_2x_4 + x_3x_6, x_3^2 - x_4^2, x_1^2 + x_4x_5).$$

It turns out that $\mathfrak{a}$ is a complete intersection. In particular, $H^2_{\mathfrak{m}}(R/\mathfrak{a})_0 = 0$ because $R/\mathfrak{a}$ is a 3-dimensional Cohen–Macaulay ring. However, using the software *Macaulay2* [GS], one sees that $H^2_{\mathfrak{m}}(R/\mathfrak{b})_0 \cong \mathrm{Ext}^4_S(R/\mathfrak{b}, R)_{-6}$ is a 1-dimensional $\mathbb{Q}$-vector space, where

$$\mathfrak{b} = \sqrt{\mathfrak{a}} = (x_2x_4 + x_3x_6, x_3^2 - x_4^2, x_1^2 + x_4x_5, x_2x_3 + x_4x_6, x_1(x_2^2 - x_6^2)).$$

Notice that $\mathrm{Proj}(R/\mathfrak{b})$ is not smooth. Indeed, it is not even irreducible, although it is connected; one can check that the minimal prime ideals of $\mathfrak{b}$ are

$$\mathfrak{p}_1 = (x_3 + x_4, x_2 - x_6, x_1^2 + x_4x_5), \quad \mathfrak{p}_2 = (x_1, x_3, x_4), \quad \mathfrak{p}_3 = (x_3 - x_4, x_2 + x_6, x_1^2 + x_4x_5).$$

Notice that the same example works in every field $\Bbbk$ of characteristic 0. Indeed, upon setting $R_\Bbbk = R \otimes_{\mathbb{Q}} \Bbbk$, the $\mathfrak{p}_i R_\Bbbk$ keep on being prime ideals, $\sqrt{\mathfrak{a}R_\Bbbk} = \mathfrak{b}R_\Bbbk$, $H^2_{\mathfrak{m}R_\Bbbk}(R_\Bbbk/\mathfrak{a}R_\Bbbk)_0 = H^2_{\mathfrak{m}}(R/\mathfrak{a})_0 \otimes_{\mathbb{Q}} \Bbbk = 0$, and

$$\dim_\Bbbk H^2_{\mathfrak{m}R_\Bbbk}(R_\Bbbk/\sqrt{\mathfrak{a}R_\Bbbk})_0 = \dim_\Bbbk H^2_{\mathfrak{m}}(R/\mathfrak{b})_0 \otimes_{\mathbb{Q}} \Bbbk = 1.$$

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