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ON THE ARITHMETICAL RANK OF CERTAIN SEGRE EMBEDDINGS

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Abstract. We study the number of (set-theoretically) defining equations of Segre products of projective spaces times certain projective hypersurfaces, extending results by Singh and Walther. Meanwhile, we prove some results about the cohomological dimension of certain schemes. In particular, we solve a conjecture of Lyubeznik about an inequality involving the cohomological dimension and the étale cohomological dimension of a scheme, in the characteristic-zero-case and under a smoothness assumption. Furthermore, we show that a relationship between depth and cohomological dimension discovered by Peskine and Szpiro in positive characteristic also holds true in characteristic-zero up to dimension three.

1. Introduction

The beauty of finding the number of defining equations of a variety is expressed by Lyubeznik in [Ly2] as follows:

Part of what makes the problem about the number of defining equations so interesting is that it can be very easily stated, yet a solution, in those rare cases when it is known, usually is highly non trivial and involves a fascinating interplay of Algebra and Geometry.

In this paper we study the number of defining equations, called arithmetical rank (see Section 2), of certain Segre products of two projective varieties. Let us list some works that already exist in this direction.

(1) In their paper [BS], Bruns and Schwänzl studied the number of defining equations of a determinantal variety. In particular, they proved that the Segre product $\mathbb{P}^n \times \mathbb{P}^m \subseteq \mathbb{P}^N$, where $N = nm + n + m$, can be defined set-theoretically by $N - 2$ homogeneous equations and not less. In particular, it is a set-theoretic complete intersection if and only if $n = m = 1$.

(2) In their work [SW], Singh and Walther gave a solution in the case of $E \times \mathbb{P}^1 \subseteq \mathbb{P}^5$ where E is a smooth elliptic plane curve. The authors proved that the arithmetical rank of this Segre product is 4. Later, in [So], Song proved that the arithmetical rank of $C \times \mathbb{P}^1$, where $C \subseteq \mathbb{P}^2$ is a Fermat curve (i.e. a curve defined by the equation $x_0^d + x_1^d + x_2^d$), is 4. In particular, both $E \times \mathbb{P}^1$ and $C \times \mathbb{P}^1$ are not set-theoretic complete intersections.

In light of these results it is natural to study the following problem.

Let n, m, d be natural numbers such that $n \geq 2$ and $m, d \geq 1$, and let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d. Consider the Segre product $Z = X \times \mathbb{P}^m \subseteq \mathbb{P}^N$,

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where $N = nm + n + m$. What can we say about the number of defining equations of Z?

Notice that the arithmetical rank of Z can depend, at least a priori, by invariants different from n, m, d; in fact we will need other conditions on X. However, for certain n, m, d we can provide some answers to this question. To this aim we will use various arguments, from complex analysis to the theory of algebras with straightening law, passing through local cohomology, étale cohomology and most of commutative algebra.

In the case $n = 2$ and $m = 1$, we introduce, for every d, a locus of special smooth projective plane curves of degree d that we will denote by $\mathcal{V}_d$. This locus consists of those smooth projective curves C of degree d which have a d-flex, i.e. a point P at which the intersection multiplicity of C and the tangent line in P is equal to d. Using methods coming from "algebras with straightening law's theory" we prove that for such a curve C the arithmetical rank of the Segre product $C \times \mathbb{P}^1 \subseteq \mathbb{P}^5$ is 4, provided that $d \geq 3$ (see Corollary 3.5). It is easy to show that every smooth elliptic curve belongs to $\mathcal{V}_3$ and that every Fermat curve of degree d belongs to $\mathcal{V}_d$, so we recover the results obtained in [SW] and in [So]. However, the equations that we will find are different from those found in these papers, and our result is characteristic free. Note that a result of Casnati and Del Centina [CD] shows that the codimension of $\mathcal{V}_d$ in the locus of all the smooth projective plane curves of degree d is $d - 3$, provided that $d \geq 3$ (Remark 3.6).

For a general n, we can prove that if $X \subseteq \mathbb{P}^n$ is a general smooth hypersurface of degree not larger than $2n - 1$, then the arithmetical rank of $X \times \mathbb{P}^1 \subseteq \mathbb{P}^{2n+1}$ is at most $2n$ (Corollary 3.8). To establish this we need a higher-dimensional version of $\mathcal{V}_d$. This result is somehow in the direction of the open question of whether any connected projective scheme of positive dimension in $\mathbb{P}^N$ can be defined set-theoretically by $N - 1$ equations.

With some similar tools we can show that, if $F = x_n^d + \sum_{i=0}^{n-3} x_i G_i(x_0, \ldots, x_n)$ and $X = \mathcal{V}_+(F)$ is smooth, then the arithmetical rank of $X \times \mathbb{P}^1 \subseteq \mathbb{P}^{2n+1}$ is $2n - 1$ (Theorem 3.9).

Using techniques similar to those of [SW], we are able to show the following: the arithmetical rank of the Segre product $C \times \mathbb{P}^m \subseteq \mathbb{P}^{3m+2}$, where C is a smooth conic of $\mathbb{P}^2$, is equal to $3m$, provided that $\mathrm{char}(k) \neq 2$ (Theorem 3.11). In particular, $C \times \mathbb{P}^m$ is a set-theoretic complete intersection if and only if $m = 1$.

Lower bounds for the arithmetical rank usually come from cohomological considerations. We collect the necessary ingredients in Section 2 using results from papers of Hartshorne [Har2], of Ogus [Og] and of Lyubeznik [Ly3] regarding the cohomological dimension of open subschemes of projective schemes. Actually, to our purpose we could use only étale cohomology. In fact the results obtained in Subsection 2.2 are sufficient to compute the number of defining equations of the varieties described above. However, when the characteristic of the base field is 0, it is possible to get the same lower bound (also in a more general setting) by reducing to the case when $k = \mathbb{C}$ and using singular, local and sheaves cohomology.

The results of Section 2 yield some nice consequences, independent from Section 3:

(1) For any n, m and d, if X is smooth, the arithmetical rank of $X \times \mathbb{P}^m \subseteq \mathbb{P}^N$ can vary just among $N - 2$, $N - 1$ and N.
(2) A conjecture of Lyubeznik in [Ly4] (see Conjecture 2.20) states, roughly speaking, that "the ètale cohomology provides a better lower bound for the

arithmetical rank than the local cohomology". We prove the conjecture in the characteristic 0 case under a smoothness assumption; see Theorem 2.21.

(3) We extend a result of Speiser obtained in characteristic 0 in [Sp], regarding the arithmetical rank of the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$ to any characteristic; see Corollary 2.14.

(4) As a consequence of Theorem 2.18, we get that if a smooth projective surface X has a Cohen-Macaulay homogeneous coordinate ring, then the cohomological dimension of its complement in any $\mathbb{P}^n$ is the least possible $(\operatorname{codim}_{\mathbb{P}^n} X - 1)$. In positive characteristic the analog version was proved in any dimension by Peskine and Szpiro in [PS]. Instead, in characteristic 0 the statement already fails for threefolds. This fact raises a nice question about a relationship between depth and cohomological dimension (Question 2.19).

2. Preliminaries for the lower bounds

As already said in the introduction, in this section we will get the necessary lower bounds we need by using results about the cohomological dimension of open subschemes of projective schemes.

First we describe in a precise way the setting in which we will work. For a Noetherian ring R and an ideal $I \subseteq R$ we define the arithmetical rank of I with respect to R as the integer

$$\operatorname{ara}(I) = \min\{k : \exists\, f_1, \ldots, f_k \in R \text{ such that } \sqrt{I} = \sqrt{(f_1, \ldots, f_k)}\}.$$

Notice that to be more precise we should write $\operatorname{ara}_R(I)$; however, it will always be clear from the context who is R. A lower bound for the arithmetical rank is given by Krull's Hauptidealsatz:

$$\operatorname{ara}(I) \geq \operatorname{ht}(I).$$

If R is graded and I homogeneous, we can also define the homogeneous arithmetical rank, that is, the integer

$$\operatorname{ara}_h(I) = \min\{k : \exists\, f_1, \ldots, f_k \in R \text{ homogeneous such that } \sqrt{I} = \sqrt{(f_1, \ldots, f_k)}\}.$$

Obviously we have

$$\operatorname{ara}(I) \leq \operatorname{ara}_h(I).$$

Assume that R is a polynomial ring of $N+1$ variables over a field k and that I is a homogeneous ideal of R. Then $\operatorname{ara}(I)$ gives the least number of hypersurfaces of the affine space $\mathbb{A}^{N+1}$ to set-theoretically define $\mathcal{V}(I) = \{\wp \in \operatorname{Spec}(R) \ : \ \wp \supseteq I\} \subseteq \mathbb{A}^{N+1} = \operatorname{Spec}(R)$. Similarly, $\operatorname{ara}_h(I)$ gives the least number of hypersurfaces of $\mathbb{P}^N$ to intersect set-theoretically to obtain $\mathcal{V}_+(I) = \{\wp \in \operatorname{Proj}(R) \ : \ \wp \supseteq I\} \subseteq \mathbb{P}^N = \operatorname{Proj}(R)$. It is an open problem whether these two numbers are always equal (see the survey article of Lyubeznik [Ly1]).

Remark 2.1. The reader should be careful of the following: the number $\operatorname{ara}(I)$, where I is an ideal of a polynomial ring, in general, does not give the minimal number of polynomials whose zero-locus is the same zero-locus of I, namely $\mathcal{Z}(I)$. For instance, if $I = (f_1, \ldots, f_m) \subseteq \mathbb{R}[x_0, \ldots, x_N]$, then clearly

$$\mathcal{Z}(I) = \mathcal{Z}(f_1^2 + \ldots + f_m^2).$$

However, $\operatorname{ara}(I)$ can be larger than 1. The reader should keep in mind that, unless the base field is algebraically closed, there is no relation between $\mathcal{V}(I)$ and $\mathcal{Z}(I)$.

We will say that I (or $X = \mathcal{V}_+(I)$) is a *set-theoretic complete intersection* if $\operatorname{ara}_h(I) = \operatorname{ht}(I) = \operatorname{codim}_{\mathbb{P}^N} X$.

For a Noetherian ring R and an ideal $I \subseteq R$ the cohomological dimension $\operatorname{cd}(R, I)$ of I (with respect to R) is the supremum of the integers i such that there exists an R-module M for which $H^i_I(M) \neq 0$. It is well known that

$$\operatorname{ara}(I) \geq \operatorname{cd}(R, I) \geq \operatorname{ht}(I).$$

In the same way, the cohomological dimension $\operatorname{cd}(X)$ of a scheme X is the supremum integer i such that there exists a quasi-coherent sheaf $\mathcal{F}$ such that $H^i(X, \mathcal{F}) \neq 0$.

If R is a finitely generated positively graded k-algebra and $I \subseteq R$ is a homogeneous ideal non-nilpotent, then

$$\operatorname{cd}(R, I) - 1 = \operatorname{cd}(\operatorname{Spec}(R) \setminus \mathcal{V}(I)) = \operatorname{cd}(\operatorname{Proj}(R) \setminus \mathcal{V}_+(I)) \tag{1}$$

(see Hartshorne [Har1]). So to bound the arithmetical rank of I, and hence the homogeneous arithmetical rank, we will give bounds on $\operatorname{cd}(\operatorname{Proj}(R) \setminus \mathcal{V}_+(I))$.

2.1. **Bounds in characteristic 0.** Throughout this subsection k (or K) will denote a field of characteristic 0. The following remark allows us to assume, in many cases, that the base field is $\mathbb{C}$.

Remark 2.2. Let R be an A-algebra, $\mathfrak{a} \subseteq R$ an ideal, B a flat A-algebra, $R_B = R \otimes_A B$, M an R-module and $M_B = M \otimes_A B$. Using the Čech complex it is not difficult to prove that for every $j \in \mathbb{N}$,

$$H^j_{\mathfrak{a}}(M) \otimes_A B \cong H^j_{\mathfrak{a}R_B}(M_B). \tag{2}$$

Now let $S = K[x_0, \ldots, x_n]$ and $I \subseteq S$ be an ideal. Since I is finitely generated we can find a field k such that, setting $S_k = k[x_0, \ldots, x_n]$, the following properties hold:

$$k \subseteq K, \quad \mathbb{Q} \subseteq k \subseteq \mathbb{C}, \quad (I \cap S_k)S = I$$

(to this aim we only have to add to $\mathbb{Q}$ the coefficients of a set of generators of I). Since K and $\mathbb{C}$ are faithfully flat k-algebras, equation (2) implies that

$$\operatorname{cd}(S, I) = \operatorname{cd}(S_k, I \cap S_k) = \operatorname{cd}(S_{\mathbb{C}}, (I \cap S_k)S_{\mathbb{C}}), \tag{3}$$

where $S_{\mathbb{C}} = \mathbb{C}[x_1, \ldots, x_n]$.

In the above situation assume that I is homogeneous and that $X = \operatorname{Proj} S/I$ is smooth over K. Then set $X_k = \operatorname{Proj}(S_k/(I \cap S_k))$ and $X_{\mathbb{C}} = \operatorname{Proj}(S_{\mathbb{C}}/((I \cap S_k)\mathbb{C}))$. Notice that $X \cong X_k \times_k \operatorname{Spec} K$, $X_{\mathbb{C}} \cong X_k \times_k \operatorname{Spec} \mathbb{C}$, and that X_k (respectively $X_{\mathbb{C}}$) is smooth over k (respectively over $\mathbb{C}$). By base change (see Liu [Li, Chapter 6, Proposition 1.24 (a)]) and by the fact that K and $\mathbb{C}$ are both flat k-algebras, we get, for all natural numbers i, j,

$$H^i(X, \Omega^j_{X/K}) \cong H^i(X_k, \Omega^j_{X_k/k}) \otimes_k K$$

and

$$H^i(X_{\mathbb{C}}, \Omega^j_{X_{\mathbb{C}}/\mathbb{C}}) \cong H^i(X_k, \Omega^j_{X_k/k}) \otimes_k \mathbb{C}$$

(see [Li, Chapter 5, Proposition 2.27]). Particularly, we have

$$\dim_K(H^i(X, \Omega^j_{X/K})) = \dim_{\mathbb{C}}(H^i(X_{\mathbb{C}}, \Omega^j_{X_{\mathbb{C}}/\mathbb{C}})). \tag{4}$$

In the rest of this subsection k will denote a field of characteristic 0. Moreover, if X is a projective variety smooth over k, we will write $h^{ij}(X)$ for $\dim_k(H^i(X,\Omega^j_{X/k}))$.

In the next remark, for the convenience of the reader, we collect some well-known facts which we will use throughout the paper.

Remark 2.3. Let X be a projective scheme over $\mathbb{C}$. We will denote by $\beta_i(X)$ the topological Betti number

$$\begin{aligned}\beta_i(X) &= \operatorname{rank}_{\mathbb{Z}}(H_i^{Sing}(X_{an},\mathbb{Z})) = \operatorname{rank}_{\mathbb{Z}}(H^i_{Sing}(X_{an},\mathbb{Z}))\\ &= \dim_{\mathbb{C}}(H^i_{Sing}(X_{an},\mathbb{C})) = \dim_{\mathbb{C}}(H^i(X_{an},\underline{\mathbb{C}}))\end{aligned}$$

(X_{an} means X regarded as an analytic space in the sense of Serre [Se], and $\underline{\mathbb{C}}$ denotes the locally constant sheaf associated to $\mathbb{C}$). Pick another projective scheme over $\mathbb{C}$, say Y, and denote by Z the Segre product $X\times Y$. The Künneth formula for singular cohomology (for instance see Hatcher [Hat, Theorem 3.16]) yields

$$H^i_{Sing}(Z_{an},\mathbb{C}) \cong \bigoplus_{p+q=i} H^p_{Sing}(X_{an},\mathbb{C}) \otimes_{\mathbb{C}} H^q_{Sing}(Y_{an},\mathbb{C});$$

thus

$$\beta_i(Z) = \sum_{p+q=i} \beta_p(X)\beta_q(Y). \tag{5}$$

Now assume that X is a projective variety smooth over $\mathbb{C}$. It is well known that X_{an} is a Kähler manifold, so the Hodge decomposition (see the notes of Arapura [Ar, Theorem 10.2.4]) is available. Therefore, together with a theorem of Serre (see [Se, Theoreme 1, pag. 19]) we have

$$H^i_{Sing}(X_{an},\mathbb{C}) \cong \bigoplus_{p+q=i} H^p(X_{an},(\Omega_{X/\mathbb{C}})^q_{an}) \cong \bigoplus_{p+q=i} H^p(X,\Omega^q_{X/\mathbb{C}}),$$

where $\mathcal{F}_{an}$ is the analyticization of a sheaf $\mathcal{F}$ (see [Se]). Thus

$$\beta_i(X) = \sum_{p+q=i} h^{pq}(X). \tag{6}$$

Finally, note that the restriction map on singular cohomology,

$$H^i_{Sing}(\mathbb{P}^n_{an},\mathbb{C}) \longrightarrow H^i_{Sing}(X_{an},\mathbb{C}), \tag{7}$$

is injective provided that $i = 0,\dots,2\dim X$ (see Shafarevich [Sh, pp. 121-122]). In particular, since $\beta_{2i}(\mathbb{P}^n) = 1$ if $i \le n$, it follows that

$$\beta_{2i}(X) \ge 1 \quad \text{provided that } i \le \dim X. \tag{8}$$

The following theorem is a quite simple consequence of the results of [Og]. It provides some necessary and sufficient conditions for the cohomological dimension of the complement of a smooth variety in a projective space to be smaller than a given integer.

Theorem 2.4. *Let $X \subseteq \mathbb{P}^n$ be a projective variety smooth over k, r an integer greater than or equal to $\operatorname{codim}_{\mathbb{P}^n} X$ and $U = \mathbb{P}^n \setminus X$. Then $\operatorname{cd}(U) < r$ if and only if*

$$h^{pq}(X) = \begin{cases} 0 & \text{if } p \ne q,\ p+q < n-r,\\ 1 & \text{if } p = q,\ p+q < n-r.\end{cases}$$

Moreover, if $k = \mathbb{C}$, the above conditions are equivalent to

$$\beta_i(X) = \begin{cases} 1 & \text{if } i < n-r \text{ and } i \text{ is even,} \\ 0 & \text{if } i < n-r \text{ and } i \text{ is odd.} \end{cases}$$

Proof. By equations (3) and (4) of Remark 2.2 we can reduce the problem in the case in which $k = \mathbb{C}$. So the "only if" part follows by a result of Hartshorne [Har2, Corollary 7.5, p. 148].

So it remains to prove the "if" part. By a theorem of Grothendieck in [Gr1] algebraic De Rham cohomology agrees with singular cohomology. Therefore, by the last part of Remark 2.3 the restriction maps

$$H^i_{DR}(\mathbb{P}^n) \longrightarrow H^i_{DR}(X) \tag{9}$$

(where H_{DR} denotes the algebraic De Rham cohomology) are injective for all $i \leq 2 \dim X$. By the assumptions, equation (6) yields $\beta_i(X) = 1$ if i is even and $i < n-r$, 0 otherwise. Moreover, $\beta_i(\mathbb{P}^n) = 1$ if i is even and $i \leq 2n$, 0 otherwise. So again using the result of Grothendieck the maps in (9) are isomorphisms for all $i < n-r$.

Now we would use a result of Ogus ([Og, Theorem 4.4]), and to this aim we will show that the De Rham depth of X is greater than or equal to $n-r$. By the proof of [Og, Theorem 4.1] this is equivalent to the fact that $\operatorname{Supp}(H^i_{\mathfrak{a}}(S)) \subseteq \mathfrak{m}$ for all $i > r$, where $S = \mathbb{C}[x_0, \ldots, x_n]$, $\mathfrak{a} \subseteq S$ is the ideal defining X and $\mathfrak{m}$ is the maximal irrelevant ideal of S. But this is easy to see, because if $\wp$ is a graded prime ideal containing $\mathfrak{a}$ and different from $\mathfrak{m}$, X being non-singular, $\mathfrak{a}S_\wp$ is a complete intersection in $S_\wp$. So $(H^i_{\mathfrak{a}}(S))_\wp \cong H^i_{\mathfrak{a}S_\wp}(S_\wp) = 0$ for all $i > r$ ($\geq \operatorname{ht}(\mathfrak{a}S_\wp)$). Hence [Og, Theorem 4.4] yields the conclusion.

Finally, if $k = \mathbb{C}$, the last condition is a consequence of the first one by equation (6). Moreover, it implies the first one because the restriction maps of singular cohomology

$$H^i_{Sing}(\mathbb{P}^n_{an}, \mathbb{C}) \longrightarrow H^i_{Sing}(X_{an}, \mathbb{C})$$

(that are injective if $i < n-r$ by the last part of Remark 2.3) are compatible with the Hodge decomposition (see [Ar, Corollary 11.2.5]). □

Remark 2.5. Theorem 2.4 does not hold in positive characteristic: for instance, pick an elliptic curve E over a field of positive characteristic whose Hasse invariant is 0. Then set $X = E \times \mathbb{P}^1 \subseteq \mathbb{P}^5$ and $U = \mathbb{P}^5 \setminus X$. The Frobenius acts as 0 on $H^1(X, \mathcal{O}_X)$, so $\operatorname{cd}(U) = 2$ (see Hartshorne and Speiser [HS] or Lyubeznik [Ly5]). However, notice that $H^1(X, \mathcal{O}_X) \neq 0$.

The two propositions below provide the necessary lower bound we need to compute the arithmetical rank of certain Segre products in characteristic 0.

Proposition 2.6. *Let X and Y be two positive-dimensional projective schemes smooth over k, and set $Z = X \times Y \subseteq \mathbb{P}^N$ (any embedding) and $U = \mathbb{P}^N \setminus Z$. Then $\operatorname{cd}(U) \geq N-3$. In particular, if $\dim Z \geq 3$, Z is not a set-theoretic complete intersection.*

Proof. By equation (3) we can assume $k = \mathbb{C}$. Using equation (8) we have $\beta_0(X) \geq 1$, $\beta_2(X) \geq 1$, $\beta_0(Y) \geq 1$ and $\beta_2(Y) \geq 1$, so equation (5) implies $\beta_2(Z) \geq 2$. Now equation (6) and Theorem 2.4 yield the conclusion. □

Remark 2.7. The proof of Proposition 2.6 yields the following nice fact.

Let X and Y be two positive-dimensional projective varieties smooth over $\mathbb{C}$ and $Z = X \times Y \subseteq \mathbb{P}^N$. Then the dimension of the secant variety of Z in $\mathbb{P}^N$ is at least $2 \dim Z - 1$.

To prove this, note, as in the proof of Proposition 2.6, that $\beta_2(Z) \geq 2$. By a theorem of Barth (see Lazarsfeld [La, Theorem 3.2.1]), it follows that Z cannot be embedded in any $\mathbb{P}^M$ with $M < 2 \dim X - 1$. If the dimension of the secant variety were less than $2 \dim X - 1$, it would be possible to project down in a biregular way X from $\mathbb{P}^N$ in $\mathbb{P}^{2 \dim X - 2}$, and this would be a contradiction.

Note that the above lower bound is the best possible; in fact, $\mathbb{P}^r \times \mathbb{P}^s$ can be embedded in $\mathbb{P}^{2(r+s)-1}$ (see Hartshorne [Har3, p. 1026]).

Remark 2.8. The statement of Proposition 2.6 is false in positive characteristic. To see this, consider two Cohen-Macaulay graded k-algebras A and B of negative a-invariant. Set $R = A\#B$ their Segre product (with the notation of the paper of Goto and Watanabe [GW]). By [GW, Theorem 4.2.3] R is Cohen-Macaulay as well. So, presenting R as a quotient of a polynomial ring of $N+1$ variables, say $R \cong P/I$, a theorem of Peskine and Szpiro in [PS] implies that $\operatorname{cd}(P, I) = N + 1 - \dim R$ (because $\operatorname{char}(k) > 0$). Translating in the language of Proposition 2.6 we have $X = \operatorname{Proj}(A)$, $Y = \operatorname{Proj}(B)$, $Z = \operatorname{Proj}(R) \subseteq \mathbb{P}^N = \operatorname{Proj}(P)$ and $\operatorname{cd}(\mathbb{P}^N \setminus Z) = \operatorname{cd}(P, I) - 1 = N - \dim Z - 1$.

Proposition 2.9. *Assume that X is a projective variety smooth over k such that $H^1(X, \mathcal{O}_X) \neq 0$ and let Y be any projective scheme over k. As above set $Z = X \times Y \subseteq \mathbb{P}^N$ (any embedding) and $U = \mathbb{P}^N \setminus Z$. Then $\operatorname{cd}(U) \geq N - 2$.*

Proof. By virtue of Remark 2.2 we may assume $k = \mathbb{C}$. The assumptions imply that $\beta_1(X) \neq 0$ by equation (6), and so $\beta_1(Z) \neq 0$ by equation (5). Clearly U is smooth, so [Har2, Theorem 7.4, p. 148] implies the conclusion. □

Remark 2.10. If in the situation of Proposition 2.9 $\dim Z \geq 2$, then it follows that Z cannot be a set–theoretical complete intersection. This is a consequence of a more general result of Hartshorne obtained in [Har1], which states that an irregular projective variety X over a field of characteristic 0 (i.e. $q(X) = h^{10}(X) \neq 0$), of dimension greater than 1, cannot be a set-theoretical complete intersection in any $\mathbb{P}^n$.

2.2. **Bounds in arbitrary characteristic.** If the characteristic of the base field is 0, we have seen in the previous subsection that we can, usually, reduce the problem to $k = \mathbb{C}$. In this context is available the complex topology, so we can use methods from algebraic topology and from complex analysis.

Unfortunately, when the characteristic of k is positive, the above techniques are not available. Moreover, some of the results obtained in Subsection 2.1 are not true in positive characteristic, as we have shown in Remarks 2.5 and 2.8. To avoid these difficulties we have to use étale cohomology, which gives a lower bound for the number of equations defining a variety as well as local cohomology (see equation (10) of Remark 2.11). This subject was introduced by Grothendieck in [Gr3]. Other references are the book [Mi1] and the lectures [Mi2] of Milne.

For a scheme X we denote by $X_{\acute{e}t}$ the étale site of X and, with a slight abuse of notation, by $\mathbb{Z}/l\mathbb{Z}$ the constant sheaf associated to $\mathbb{Z}/l\mathbb{Z}$ (for any $l \in \mathbb{Z}$). Moreover, we denote by $\acute{e}\operatorname{cd}(X)$ the étale cohomological dimension of X; that is, the largest integer i such that there exists a torsion sheaf $\mathcal{F}$ on $X_{\acute{e}t}$ with étale cohomology

group $H^i(X_{\acute{e}t}, \mathcal{F}) \neq 0$ (H^i denotes the usual cohomology of sheaves). Below we collect some basic results about the étale cohomological dimension.

Remark 2.11. If X is an n-dimensional scheme of finite type over a separably closed field, then $\text{écd}(X) \leq 2n$ ([Mi1, Chapter VI, Theorem 1.1]). If moreover X is affine, then $\text{écd}(X) \leq n$ ([Mi1, Chapter VI, Theorem 7.2]).

Assume that $X = \text{Proj}(R)$ is projective and pick a closed subscheme $Y = \mathcal{V}_+(I) \subseteq X$. Then $U = X \setminus Y$ can be covered by $\text{ara}_h(I)$ affine subsets of X. Moreover, the étale cohomological dimension of these affine subsets is less than or equal to n by what is said above. So, using repetitively the Mayer-Vietoris sequence ([Mi1, Chapter III, Exercise 2.24]), it is easy to prove that

$$\text{écd}(U) \leq n + \text{ara}_h(I) - 1. \tag{10}$$

The above inequality was remarked, for instance, by Newstead in [Ne].

We recall the following result of [Ly3, Proposition 9.1, (iii)], which can be seen as an étale version of [Har2, Theorem 8.6, p. 160].

Theorem 2.12 (Lyubeznik)**.** *Let k be a separably closed field of arbitrary characteristic, and let $Y \subseteq X$ be two projective varieties such that $U = X \setminus Y$ is non-singular. Set $N = \dim X$, and $l \in \mathbb{Z}$ coprime with the characteristic of k. If $\text{écd}(U) < 2N - r$, then the restriction maps*

$$H^i(X_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) \longrightarrow H^i(Y_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z})$$

are isomorphisms for $i < r$ and injective for $i = r$.

Remark 2.13. The étale version of Theorem 2.4 does not hold. In fact, the integer $\text{écd}(\mathbb{P}^N \setminus Y)$ is not an invariant of only Y and N (as instead is for the integer $\text{cd}(\mathbb{P}^N \setminus Y)$). For instance, we can consider $\mathbb{P}^2 \subseteq \mathbb{P}^5$ (embedded as a linear subspace) and $v_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ (where $v_2(\mathbb{P}^2)$ is the 2nd Veronese embedding): the first one is defined (also scheme-theoretically) by 3 linear equations, so $\text{écd}(\mathbb{P}^5 \setminus \mathbb{P}^2) \leq 7$ by equation (10); instead, $\text{écd}(\mathbb{P}^5 \setminus v_2(\mathbb{P}^2)) = 8$ by [Bar].

Notice that the above argument shows that the number of defining equations of a projective scheme $X \subseteq \mathbb{P}^n$ depends on the embedding, and not only on X and on n. This suggests the limits of the use of local cohomology on certain problems regarding the arithmetical rank.

In [Sp] Speiser, among other things, computed the arithmetical rank of the diagonal $\Delta = \Delta(\mathbb{P}^n) \subseteq \mathbb{P}^n \times \mathbb{P}^n$, provided that the characteristic of the base field is 0. In characteristic $p > 0$ he proved that the cohomological dimension of $\mathbb{P}^n \times \mathbb{P}^n \setminus \Delta$ is the least possible, i.e. $n - 1$, therefore in positive characteristic the arithmetical rank of Δ is not known. Actually Theorem 2.12 easily implies that the result of Speiser holds in arbitrary characteristic, since the upper bound found in [Sp] is valid in arbitrary characteristic. However, since in that paper the author did not describe the equations needed to define Δ set-theoretically, we provide the upper bound with a different method, that yields an explicit set of equations for Δ.

To this aim, we recall that the coordinate ring of $\mathbb{P}^n \times \mathbb{P}^n$ is $A = k[x_i y_j : i, j = 0, \ldots, n]$ and the ideal $I \subseteq A$ defining Δ is $I = (x_i y_j - x_j y_i : 0 \leq i < j \leq n)$.

Corollary 2.14. *In the situation described above (with k a separably closed field of arbitrary characteristic)* $\text{ara}_h(I) = 2n - 1$.

Proof. As has been previously said, by [Sp, Proposition 2.1.1] we already know that $\operatorname{ara}_h(I) \leq 2n-1$. However, we can observe that if we consider $IR \subseteq R = k[x_i, y_j : i,j = 0,\dots,n]$, then IR is the ideal generated by the 2-minors of the $2 \times (n+1)$ matrix of indeterminates whose rows are, respectively, $x_0,\dots,x_n$ and $y_0,\dots,y_n$. So, by [BrVe, (5.9) Lemma], a set of generators of $IR \subseteq R$ up to radical is

$$g_k = \sum_{\substack{0 \leq i < j \leq n \\ i+j=k}} (x_i y_j - x_j y_i), \quad k = 1,\dots,2n-1.$$

Since these polynomials belong to A and since A is a direct summand of R, we get

$$\sqrt{(g_1,\dots,g_{2n-1})A} = I.$$

Therefore $\operatorname{ara}_h(I) \leq 2n-1$.

For the lower bound choose l coprime with $\operatorname{char}(k)$. The Künneth formula for étale cohomology ([Mi1, Chapter VI, Corollary 8.13]) implies that

$$H^2(\mathbb{P}^n_{\acute{e}t} \times \mathbb{P}^n_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) \cong (\mathbb{Z}/l\mathbb{Z})^2,$$

while $H^2(\Delta_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) \cong H^2(\mathbb{P}^n_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) \cong \mathbb{Z}/l\mathbb{Z}$. So Theorem 2.12 yields $\operatorname{écd}(U) \geq 4n-2$, where $U = \mathbb{P}^n \times \mathbb{P}^n \setminus \Delta$. Therefore equation (10) yields the conclusion. □

The next two propositions are the analogue of Propositions 2.6 and 2.9. We need them to compute the homogeneous arithmetical rank of certain Segre products in arbitrary characteristic. First we need a remark:

Remark 2.15. Let X be a projective variety smooth over a field k and l an integer coprime to $\operatorname{char}(k)$. The kernel of the cycle map is contained in the kernel of the projection from the Chow ring to itself modulo numerical equivalence. But this last group is non-zero because X is projective, so we have

$$H^{2i}(X_{\acute{e}t}, \mathbb{Z}_l) \neq 0 \quad \forall\, i = 0,\dots,\dim X. \tag{11}$$

Therefore there exists an integer n such that $H^{2i}(X_{\acute{e}t}, \mathbb{Z}/l^n\mathbb{Z})$ is non-zero for any $i = 0,\dots,\dim X$. See the proof of [Mi1, Chapter VI, Theorem 11.7].

Proposition 2.16. *Let k be an algebraic closed field of arbitrary characteristic. Let X and Y be two projective varieties smooth over k of dimension at least 1. Set $Z = X \times Y \subseteq \mathbb{P}^N$ (any embedding) and $U = \mathbb{P}^N \setminus Z$. Then $\operatorname{écd}(U) \geq 2N-3$. In particular, if $\dim Z \geq 3$, Z is not a set-theoretic complete intersection.*

Proof. By the above remark there is an integer l coprime with $\operatorname{char}(k)$ such that the modules $H^i(X_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z})$ and $H^i(Y_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z})$ are non-zero $\mathbb{Z}/l\mathbb{Z}$-modules. But $H^2(\mathbb{P}^N_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) \cong \mathbb{Z}/l\mathbb{Z}$, therefore by the Künneth formula for étale cohomology ([Mi1, Chapter VI, Corollary 8.13]) it follows that $H^2(Z_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z})$ cannot be isomorphic to $H^2(\mathbb{P}^N_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z})$. Now Theorem 2.12 implies the conclusion. □

Proposition 2.17. *Let k be an algebraically closed field, C a smooth projective curve of positive genus, X a projective scheme and $Y = C \times X \subseteq \mathbb{P}^N$ (any embedding). Then $\operatorname{écd}(\mathbb{P}^N \setminus Y) \geq 2N-2$. In particular, if $\dim X \geq 1$, then Y is not a set-theoretic complete intersection.*

Proof. Set g the genus of C. By [Mi2, Proposition 14.2 and Remark 14.4] it follows that $H^1(C_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) \cong (\mathbb{Z}/l\mathbb{Z})^{2g}$. Moreover, $H^0(X_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) \neq 0$ and $H^1(\mathbb{P}^N_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) = 0$. But by the Künneth formula for étale cohomology $H^1(Y_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) \neq 0$, therefore we conclude by Theorem 2.12. □

2.3. **Consequences on the cohomological dimension.** In this subsection we draw two nice consequences of the investigations we made in the first part of the work. They are in the direction of a problem stated by Grothendieck, who asked in [Gr2, p. 79] to find conditions, fixed a positive integer c, under which $\operatorname{cd}(R,I) \leq c$, where I is an ideal in a ring R.

The first fact we want to present is a consequence of Theorem 2.4, and it regards a relationship between cohomological dimension of an ideal in a polynomial ring and the depth of the relative quotient ring. It was proved by Peskine and Szpiro in [PS] that if $I \subseteq S = k[x_1, \ldots, x_n]$ is a homogeneous ideal of a polynomial ring over a field of positive characteristic such that $\operatorname{depth}(S/I) \geq t$, then $\operatorname{cd}(S,I) \leq n-t$. The same assertion does not hold in characteristic 0; in fact, examples are already known for $t = 4$ (for instance if I defines the Segre product of two projective spaces). When $t = 2$ the statement is also true in characteristic 0 by a result of Hartshorne and Speiser (for instance see [Har2]). We can settle the case $t = 3$ in the smooth case.

Theorem 2.18. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field of characteristic 0. If $I \subseteq S$ is a homogeneous prime ideal such that $(S/I)_{\wp}$ is a regular local ring for any homogeneous prime ideal $\wp \neq \mathfrak{m} = (x_1, \ldots, x_n)$ and such that $\operatorname{depth}(S/I) \geq 3$, then $\operatorname{cd}(S,I) \leq n-3$.*

Proof. Suppose by contradiction that $\operatorname{cd}(S,I) \geq n-2$. Set $X = \operatorname{Proj}(S/I) \subseteq \mathbb{P}^{n-1} = \operatorname{Proj}(S)$. So we are supposing that $\operatorname{cd}(\mathbb{P}^{n-1} \setminus X) \geq n-3$ by equation (1). By the assumptions X is a projective variety smooth over k. Therefore Theorem 2.4 implies that $h^{10}(X) \neq 0$ or that $h^{01}(X) \neq 0$. But with the notation of Remark 2.2, $h^{10}(X) = h^{10}(X_{\mathbb{C}})$ and $h^{01}(X) = h^{01}(X_{\mathbb{C}})$. So, since $h^{10}(X_{\mathbb{C}}) = h^{01}(X_{\mathbb{C}})$ (using [Ar, Theorem 10.2.4] and [Se] together), we have $h^{10}(X) \neq 0$. But $H^1(X, \mathcal{O}_X) = [H^2_{\mathfrak{m}}(S/I)]_0 \subseteq H^2_{\mathfrak{m}}(S/I)$ ($[\ ]_0$ denotes the 0-degree part), so $\operatorname{depth}(S/I) \leq 2$, which is a contradiction. □

Actually, the cited result of Peskine and Szpiro holds true whenever the ambient is a regular local ring of positive characteristic. Moreover, one can easily deduce by the result of Huneke and Lyubeznik [HL, Theorem 2.9] the following: If R is an n-dimensional regular local ring containing its residue field and $\mathfrak{a} \subseteq R$ is an ideal such that $\operatorname{depth}(R/\mathfrak{a}) \geq 2$, then $\operatorname{cd}(R,\mathfrak{a}) \leq n-2$. Together with these facts, Theorem 2.18 raises the following question:

Question 2.19. Suppose that R is a regular local ring and that $I \subseteq R$ is an ideal such that $\operatorname{depth}(R/I) \geq 3$. Is it true that $\operatorname{cd}(R,I) \leq \dim R - 3$?

The second fact we want to show is a consequence of Theorem 2.4 and Theorem 2.12. It provides a solution of a special case of a conjecture stated by Lyubeznik in [Ly4, Conjecture, p. 147]:

Conjecture 2.20 (Lyubeznik)**.** *If U is an n-dimensional scheme of finite type over a separably closed field, then $\operatorname{écd}(U) \geq n + \operatorname{cd}(U)$.*

Theorem 2.21. *Let $X \subseteq \mathbb{P}^n$ be a projective variety smooth over $\mathbb{C}$ and let $U = \mathbb{P}^n \setminus X$. Then*

$$\operatorname{écd}(U) \geq n + \operatorname{cd}(U).$$

Proof. Set $\operatorname{cd}(U) = s$ and define an integer ρ_s to be 0 (resp. 1) if $n-s-1$ is odd (resp. if $n-s-1$ is even). By Theorem 2.4 and equation (8), it follows

that $\beta_{n-s-1}(X) > \rho_s$. Consider, for a prime number p, the $\mathbb{Z}/p\mathbb{Z}$-vector space $\mathrm{Hom}_{\mathbb{Z}}(H_i^{Sing}(X_{an},\mathbb{Z}),\mathbb{Z}/p\mathbb{Z})$. Since $H_i^{Sing}(X_{an},\mathbb{Z})$ is of rank larger than ρ_s, the above $\mathbb{Z}/p\mathbb{Z}$-vector space has dimension greater than ρ_s. Therefore by the surjection given by the universal coefficient theorem,

$$H_{Sing}^{n-s-1}(X_{an},\mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_{n-s-1}^{Sing}(X_{an},\mathbb{Z}),\mathbb{Z}/p\mathbb{Z})$$

(see [Hat, Theorem 3.2, p. 195]), we infer that $\dim_{\mathbb{Z}/p\mathbb{Z}} H_{Sing}^{n-s-1}(X_{an},\mathbb{Z}/p\mathbb{Z}) > \rho_s$. Now a comparison theorem due to Grothendieck (see [Mi1, Chapter III, Theorem 3.12]) yields

$$\dim_{\mathbb{Z}/p\mathbb{Z}} H^{n-s-1}(X_{\acute{e}t},\mathbb{Z}/p\mathbb{Z}) > \rho_s.$$

Since $\dim_{\mathbb{Z}/p\mathbb{Z}}(H^{n-s-1}(\mathbb{P}^n_{\acute{e}t},\mathbb{Z}/p\mathbb{Z})) = \rho_s$, Theorem 2.12 implies that $\acute{\mathrm{e}}\mathrm{cd}(U) \geq 2n-(n-s) = n+s$. □

Theorem 2.21 might look like a very special case of Conjecture 2.20. However, the case when U is the complement of a projective variety in a projective space is a very important case. In fact, the truth of Conjecture 2.20 would ensure that to bound the homogeneous arithmetical rank from below it would be enough to work with just ètale cohomology and not with sheaf cohomology. Since usually one is interested in computing the number of (set-theoretically) defining equations of a projective variety in the projective space, in some sense the most interesting case of Conjecture 2.20 occurs when $U = \mathbb{P}^n \setminus X$ for some projective variety X. From this point of view, one can look at Theorem 2.21 in the following way: *In order to give a lower bound for the minimal number of hypersurfaces of* $\mathbb{P}^n_{\mathbb{C}}$ *set-theoretically cutting out a smooth projective variety* $X \subseteq \mathbb{P}^n_{\mathbb{C}}$*, it is better to compute* $\acute{\mathrm{e}}\mathrm{cd}(\mathbb{P}^n_{\mathbb{C}} \setminus X)$ *than* $\mathrm{cd}(\mathbb{P}^n_{\mathbb{C}} \setminus X)$.

Unfortunately, Lyubeznik informed the author of this paper by a personal communication that he found a counterexample, yet unpublished, to Conjecture 2.20 when the characteristic of the base field is positive. His counterexample consists of a scheme U which is the complement in $\mathbb{P}^n$ of a reducible projective scheme.

3. Upper bounds

In this section we finally present the defining equations of the varieties described in the introduction. The main tools we use come from ASL theory.

3.1. **Notation.** We want to fix some notation that we will use throughout this section. Let k be a field of arbitrary characteristic.

We recall that the Segre product of two finitely generated $\mathbb{N}$-graded k-algebras A and B is defined as

$$A\sharp B = \bigoplus_{n\in\mathbb{N}} A_n \otimes_k B_n.$$

This is an $\mathbb{N}$-graded k-algebra, and it is a direct summand of the tensor product $A \otimes_k B$.

Fix n, m integers greater than or equal to 1. Then $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ will always denote two projective schemes defined respectively by the standard graded ideals $\mathfrak{a} \subseteq R = k[x_0,\ldots,x_n]$ and $\mathfrak{b} \subseteq S = k[y_0,\ldots,y_m]$.

Consider the Segre product $Z = X \times Y$ and set $A = R/\mathfrak{a}$ and $B = S/\mathfrak{b}$. Then we have that $Z \cong \mathrm{Proj}(A\sharp B)$. Moreover, if $W := k[x_iy_j : i = 0,\ldots,n;\ \ j = 0,\ldots,m] \subseteq k[x_0,\ldots,x_n,y_0,\ldots,y_m] = R \otimes_k S$, then $A\sharp B = W/\mathcal{I}$ with $\mathcal{I} \subseteq W$ a homogeneous

ideal. Assuming that $\mathfrak{a} = (f_1, \dots, f_r)$ and $\mathfrak{b} = (g_1, \dots, g_s)$ with $\deg f_i = d_i$ and $\deg g_j = e_j$, it is easy to see that $\mathcal{I}$ is generated by the following polynomials:

(1) $M \cdot f_i$, where M varies among the monomials in S_{d_i} for every $i = 1, \dots, r$;
(2) $g_j \cdot N$, where N varies among the monomials in R_{e_j} for every $j = 1, \dots, s$.

Now we present $A \sharp B$ as a quotient of a polynomial ring. So consider $P = k[z_{ij} : i = 0, \dots, n : \ j = 0, \dots, m]$ and the k-algebra homomorphism $\phi : P \longrightarrow A \sharp B$ defined as $\phi = \phi' \circ \pi$, where $\phi' : P \longrightarrow W$ maps z_{ij} to $x_i y_j$ and $\pi : W \longrightarrow A \sharp B \cong W/\mathcal{I}$ is the projection. Therefore set $I = \operatorname{Ker} \phi$. With this notation, then, we have

$$V \cong \operatorname{Proj}(P/I) \subseteq \mathbb{P}^N, \qquad N = nm + n + m.$$

Now we describe a system of generators which we will use in this section. For all monomials $M \in S_{d_i}$ (where $i = 1, \dots, r$), choose a polynomial $(f_i)_M \in P$ such that $\phi'((f_i)_M) = M \cdot f_i$. In the same way choose a polynomial $(g_j)_N \in P$ for all monomials $N \in R_{e_j}$ and $j = 1, \dots, s$. Then it is easy to show that

$$I = I_2(Z) + J,$$

where:

(1) $I_2(Z)$ denotes the ideal generated by the 2-minors of the matrix $Z = (z_{ij})$;
(2) $J = ((f_i)_M, (g_j)_N :$ for all $i = 1, \dots, r$ and for all monomials $M \in S_{d_i}$, for all $j = 1, \dots, s$ and for all monomials $N \in R_{e_j})$.

3.2. **The defining equations of certain Segre products.** Our purpose is to exhibit a minimal set of defining equations (up to radical) for I in P, and so to compute the arithmetical rank of I. We are able to solve this problem for certain ideals $\mathfrak{a}$ and $\mathfrak{b}$.

We need the following remark to work with algebraically closed fields and to use the Nullstellensatz:

Remark 3.1. Let H be a k-algebra and $\mathfrak{h} \subseteq H$ an ideal. Set $H_{\bar{k}} = H \otimes_k \bar{k}$ and $\mathfrak{h}_{\bar{k}} = \mathfrak{h} H_{\bar{k}} \subseteq H_{\bar{k}}$, where $\bar{k}$ denotes the algebraic closure of k. Because $\bar{k}$ is faithfully flat over k, if $h_1, \dots, h_t \in \mathfrak{h}$ are such that $\sqrt{\mathfrak{h}_{\bar{k}}} = \sqrt{(h_1, \dots, h_t) H_{\bar{k}}}$, then $\sqrt{\mathfrak{h}} = \sqrt{(h_1, \dots, h_t)}$.

In the following remark we make use of an argument that will be used several times later on.

Remark 3.2. Actually the described generators of I are too much. For instance, for a polynomial f_i of the starting ideal we have to consider all the polynomials $(f_i)_M$ with M varying in S_{d_i}. These are $\binom{m+d_i}{m}$ polynomials! Anyway, up to radical, it is enough to choose $m+1$ monomials for every f_i and $n+1$ monomials for every g_j.

For every $i = 1, \dots, r$ and $l = 0, \dots, m$, set $M = y_l^{d_i}$. A possible choice for $(f_i)_M$ is $(f_i)_l := f_i(z_{0l}, \dots, z_{nl}) \in P$. In the same manner, for every $j = 1, \dots, s$ and $k = 0, \dots, n$ we define $(g_j)_k = g_j(z_{k0}, \dots, z_{km}) \in P$. We claim that

$$\sqrt{I} = \sqrt{I_2(Z) + J'},$$

where J' is the ideal generated by the $(f_i)_l$'s and the $(g_j)_k$'s.

We can assume that k is algebraically closed by Remark 3.1. So, denoting by $\mathcal{Z}(L)$ the zeroes locus of an ideal L, it is enough to prove that $\mathcal{Z}(I) = \mathcal{Z}(I_2(Z) + J')$ by Nullstellensatz. So pick $P = [P_{00}, P_{10}, \dots, P_{n0}, P_{01}, \dots, P_{n1}, \dots, P_{0m}, \dots, P_{nm}] \in \mathcal{Z}(I_2(Z) + J)$. We can write $P = [P_0, \dots, P_m]$, where $P_h = [P_{0h}, \dots, P_{nh}]$ is

$[0,0,\dots,0]$ or a point of $\mathbb{P}^n$. Since $P \in \mathcal{Z}(I_2(Z))$ it follows that the non-zero points among the P_h's are equal as points of $\mathbb{P}^n$. Moreover, if P_l is a non-zero point, $(f_i)_l(P) = 0$ for all $i = 1,\dots,r$ means that $P_l \in X$. Then from the discussion above, trivially $(f_i)_M(P) = 0$ for every i, M and any choice of $(f_i)_M$. By symmetry one can prove that all the $(g_j)_N$'s also vanish on P, so we conclude.

Remark 3.3. Assume that $X = \mathcal{V}_+(F) \subseteq \mathbb{P}^n_k$ is a projective hypersurface ($F = f_1$), $m = 1$ and $Y = \mathbb{P}^1$. We already know from a general theorem of Eisenbud and Evans (see [EE, Theorem 2]) that

$$\operatorname{ara}(I) \leq \operatorname{ara}_h(I) \leq N = 2n+1.$$

In this case we can find an explicit set of polynomials which generates I up to radical. In fact, from a theorem of Bruns and Schwánzl (see [BS, Theorem 2]), we know that

$$\operatorname{ara}(I_2(Z)) = \operatorname{ara}_h(I_2(Z)) = 2n-1.$$

Moreover, a set of homogeneous generators of $I_2(Z)$ up to radical is known. Using the notation of [BrVe], set $[i,j] = z_{i0}z_{j1} - z_{j0}z_{i1}$ for $0 \leq i < j \leq n$. Then

$$I_2(Z) = \sqrt{\Big(\sum_{i+j=k} [i,j] : k = 1,\dots,2n-1\Big)}$$

(see [BrVe, Lemma 5.9]).

By Remark 3.2 we only have to add to these generators $F_0 = (f_1)_0$ and $F_1 = (f_1)_1$ (with the notation of Remark 3.2), and so we find $2n+1$ homogeneous polynomials which generate I up to radical.

Theorem 3.4. *Let $X = \mathcal{V}_+(F) \subseteq \mathbb{P}^n$ be a hypersurface such that there exists a line $L \subseteq \mathbb{P}^n$ that meets X only at a point P, and let I be the ideal defining the Segre product $X \times \mathbb{P}^1 \subseteq \mathbb{P}^{2n+1}$. Then*

$$\operatorname{ara}_h(I) \leq 2n.$$

Proof. By a change of coordinates we can assume that $L = \mathcal{V}_+((x_0,\dots,x_{n-2}))$. The set $\Omega = \{[i,j] : i < j, i+j \leq 2n-2\}$ is an ideal of the poset of the minors of the matrix $Z = (z_{ij})$ (i.e. if $[i,j] \in \Omega$, $h \leq i$ and $h < k \leq j$, then $[h,k] \in \Omega$), so by [BrVe, Lemma 5.9],

$$\operatorname{ara}(\Omega R) \leq \operatorname{rank}(\Omega) = 2n-2.$$

We want to prove that $I = \sqrt{J}$, where $J = \Omega R + (F_0, F_1)$ (with the notation of Remarks 3.2, 3.3). To this purpose we may assume that k is algebraically closed by Remark 3.1, and we will prove the equivalent condition, by Nullstellensatz and Remark 3.2, that $\mathcal{Z}(I_2(Z) + (F_0,F_1)) = \mathcal{Z}(J)$.

Let $Q = [Q_0,Q_1] = [Q_{01},\dots,Q_{0n},Q_{10},\dots,Q_{1n}] \in \mathcal{Z}(J)$. If $Q_0 = 0$ or $Q_1 = 0$, then trivially $Q \in \mathcal{Z}(I_2(Z))$, so we assume that Q_0, Q_1 are points of $\mathbb{P}^n$. First suppose $Q_{ij} \neq 0$ for some $j \leq n-2$ and $i = 0,1$: for any $h \neq k$ different from j, $[h,j]$ (or $[j,h]$) and $[k,j]$ (or $[j,k]$) are elements of Ω, so since $Q \in \mathcal{Z}(J)$ we easily obtain the relations $Q_{0h}Q_{1k} = Q_{1h}Q_{0k}$, from which $Q \in \mathcal{Z}(I_2(Z))$. We can therefore assume that $Q_{ij} = 0$ for all $j < n-1, i = 0,1$. But then Q_0 and Q_1 belong to $L \cap X$, so $Q_0 = Q_1 = P$, so $Q \in \mathcal{Z}(I_2(Z))$. □

Corollary 3.5. *Let $X \subseteq \mathbb{P}^2$ be a smooth curve of degree $d \geq 3$ such that there exists a line $L \subseteq \mathbb{P}^2$ that meets X only at a point P, and let I be the ideal defining the Segre product $X \times \mathbb{P}^1 \subseteq \mathbb{P}^5$. Then*

$$\operatorname{ara}_h(I) = 4.$$

Moreover, if k has characteristic 0, then $\operatorname{ara}(I) = \operatorname{ara}_h(I) = 4$.

Proof. Theorem 3.4 implies that $\operatorname{ara}_h(I) \leq 4$. For the lower bound first assume that k is algebraically closed. Since X has positive genus, Proposition 2.17 implies that $\operatorname{écd}(\mathbb{P}^5 \setminus (X \times \mathbb{P}^1)) \geq 8$. Thus equation (10) of Remark 2.11 implies that $\operatorname{ara}_h(I) \geq 4$. If k is not algebraically closed, it is obvious that $\operatorname{ara}_h(I) \geq \operatorname{ara}_h(I(P \otimes_k \bar{k}))$, so we have proved the first statement.

If $\operatorname{char}(k) = 0$, Proposition 2.9 implies that $\operatorname{cd}(P, I) \geq 4$, so $\operatorname{ara}(I) \geq 4$. □

Remark 3.6. In light of Theorem 3.4 and Corollary 3.5, it is natural to define the following set. For every natural number $n, d \geq 1$ we define

$$\mathcal{V}_d^{n-1} = \{X \subseteq \mathbb{P}^n : X \text{ smooth},\ \dim X = n-1,\ \deg X = d,\ \exists\ P \in X \text{ as in 3.4}\}/\operatorname{PGL}_n(k).$$

Notice that all hypersurfaces in $\mathcal{V}_d^{n-1}$ can be represented, by a change of coordinate, by $\mathcal{V}_+(F)$ with $F = x_{n-1}^d + \sum_{i=0}^{n-2} x_i G_i(x_0, \ldots, x_n)$, where the G_i's are homogeneous polynomials of degree $d-1$.

We start to analyze the case $n = 2$, and for simplicity we will write $\mathcal{V}_d$ instead of $\mathcal{V}_d^1$. So our question is: *How many smooth projective plane curves of degree d do belong to $\mathcal{V}_d$?*

First we list some plane projective curves belonging to $\mathcal{V}_d$:

(1) Every smooth elliptic curve belongs to $\mathcal{V}_3$. In fact every smooth curve of degree greater than or equal to 3 has an ordinary flex, and an elliptic curve meets a line at most to 3 points. So we recovered [SW, Theorem 1.1] as a consequence of Corollary 3.5 (the generators up to radical are different).
(2) Obviously, every smooth conic belongs to $\mathcal{V}_2$, too.
(3) Every Fermat curve of degree d, i.e. a projective curve $C = \mathcal{V}_+(F) \subseteq \mathbb{P}^2$ where $F = x_o^d + x_1^d + x_2^d$, belongs to $\mathcal{V}_d$. In fact, one has only to consider the line $\mathcal{V}_+(x_0 + \alpha x_1)$ where $\alpha^d = -1$ and the point $[\alpha, 1, 0] \in C$, so we also recovered [So, Theorem 2.8] (the generators are again different).

In their paper [CD, Theorem A], Casnati and Del Centina compute the dimension of the loci $\mathcal{V}_{d,\alpha}$, $\alpha = 1, 2$, of all the smooth plane curves of degree d with exactly α points as in Theorem 3.4 (if these points are non-singular, as in this case, they are called d-flexes), and they showed that $\mathcal{V}_{d,\alpha}$ are irreducible rational locally closed subvarieties of the moduli space $\mathcal{M}_g$ of curve of genus $g = \binom{d-1}{2}$. The dimension of $\mathcal{V}_{d,\alpha}$ is

$$\dim(\mathcal{V}_{d,\alpha}) = \binom{d+2-\alpha}{2} - 8 + 3\alpha.$$

Moreover, it is not difficult to show that $\mathcal{V}_{d,1}$ is an open Zariski subset of $\mathcal{V}_d$ (see [CD, Lemma 2.1.2]), and so

$$\dim(\mathcal{V}_d) = \binom{d+1}{2} - 5.$$

The locus $\mathcal{H}_d$ of all smooth plane curves of degree d up to isomorphism is an open Zariski subset of $\mathbb{P}_{\mathbb{C}}^{\binom{d+2}{2}}$ modulo the group $\mathrm{PGL}_2(\mathbb{C})$, so its dimension is $\binom{d+2}{2} - 9$. Then the codimension of $\mathcal{V}_d$ in $\mathcal{H}_d$, provided $d \geq 3$, is $d-3$.

So, for example, if we pick a quartic C in the hypersurface $\mathcal{V}_4$ of $\mathcal{H}_4$, Corollary 3.5 implies that $C \times \mathbb{P}^1 \subseteq \mathbb{P}^5$ can be defined by exactly four equations. However, it remains an open problem to compute the arithmetical rank of $Y \times \mathbb{P}^1 \subseteq \mathbb{P}^5$ for any quartic $Y \subseteq \mathbb{P}^2$.

In the general case ($n \geq 2$ arbitrary) we can state the following lemma.

Lemma 3.7. *Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d. If $d \leq 2n-3$ or if $d \leq 2n-1$ and X is generic, then $X \in \mathcal{V}_d^{n-1}$.*

Proof. First we prove the following claim:

a) If $X \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree $d \leq 2n-1$ not containing lines, then $X \in \mathcal{V}_d^{n-1}$.

We denote by $\mathrm{Grass}(1,n)$ the Grassmannian of lines of $\mathbb{P}^n$. Consider the projective variety $W_n = \{(P,L) \in \mathbb{P}^n \times \mathrm{Grass}(1,n) : P \in L\}$. It turns out that this is an irreducible variety of dimension $2n-1$. Now set

$$T_{n,d} = \{((P,L),F) \in W_n \times L_{n,d} : \ i(L, \mathcal{V}_+(F), P) \geq d\},$$

where by $L_{n,d}$ we denote the projective space of all the homogeneous polynomials of degree d of $K[x_0,\ldots,x_n]$, and by $i(L,\mathcal{V}_+(F);P)$ the multiplicity intersection of L and $\mathcal{V}_+(F)$ at P (if $L \subseteq \mathcal{V}_+(F)$, then $i(L,\mathcal{V}_+(F);P) = +\infty$).

Assume that $P = [1,0,\ldots,0]$ and that L is given by the equation $x_1 = x_2 = \ldots = x_n = 0$. Then it is easy to see that for a polynomial $F \in L_{n,d}$ the condition $(P,L,F) \in T_{n,d}$ is equivalent to the fact that the coefficients of x_0^d, $x_0^{d-1}x_1$, $\ldots$, $x_0x_1^{d-1}$ in F are 0. This implies that $T_{n,d}$ is a closed subset of $\mathbb{P}^n \times \mathrm{Grass}(1,n) \times L_{n,d}$; thus $T_{n,d}$ is a projective scheme over k.

Consider the restriction of the first projection $\pi_1 : T_{n,d} \longrightarrow W_n$. Clearly π_1 is surjective; moreover, it follows by the above discussion that all the fibers of π_1 are projective subspaces of $L_{n,d}$ of dimension $\dim(L_{n,d}) - d$. Therefore $T_{n,d}$ is an irreducible projective variety of dimension $2n-1+\dim(L_{n,d})-d$.

Now consider the restriction of the second projection $\pi_2 : T_{n,d} \longrightarrow L_{n,d}$. Clearly $\mathcal{V}_+(F) \in \mathcal{V}_d^{n-1}$ provided it is smooth, it does not contain any line and it belongs to $\pi_2(T_{n,d})$. So to conclude it is enough to check the surjectivity of π_2 whenever $d \leq 2n-1$. To this aim, since both $T_{n,d}$ and $L_{n,d}$ are projective, it is enough to show that for a general $F \in \pi_2(T_{n,d})$ the dimension of the fiber $\pi_2^{-1}(F)$ is exactly $2n-1-d$. On the other hand, it is clear that the codimension of $\pi_2(T_{n,d})$ in $T_{n,d}$ is at least $d-2n+1$ when $d \geq 2n$. We proceed by induction on n (for $n=2$ we already know this).

First consider the case in which $d \leq 2n-3$. Let F be a general form of $\pi_2(T_{n,d})$, and set $r = \dim(\pi_2^{-1}(F))$. By contradiction assume that $r > 2n-1-d$. Consider a general hyperplane section of $\mathcal{V}_+(F)$, and set F' the polynomial defining it. Obviously any element of $\pi_2(T_{n-1,d})$ comes from $\pi_2(T_{n,d})$ in this way, so F' is a generic form of $\pi_2(T_{n,d})$. The condition for a line to belong to a hyperplane is of codimension 2, so the dimension of the fiber of F' is at least $r-2$. Since F' is a polynomial of $K[x_0,\ldots,x_{n-1}]$ of degree $d \leq 2(n-1)-1$, we can apply an induction getting $r-2 \leq 2n-3-d$ so that $r \leq 2n-1-d$, which is a contradiction.

We end with the case in which $d = 2n-1$ (the case $d = 2n-2$ is easier). Let F and r be as before, and suppose by contradiction that $r \geq 1$. This implies that there exists a hypersurface $\mathcal{H} \subseteq \mathrm{Grass}(n-1, n)$ such that for any general $H \in \mathcal{H}$ the polynomial defining $\mathcal{V}_+(F) \cap H$ belongs to $\pi_2(T_{n-1,d})$. This implies that the codimension of $\pi_2(T_{n-1,d})$ in $T_{n-1,d}$ is less than or equal to 1, whereas we know that this is at least 2.

So a) holds true. Now we prove the lemma by induction on n (if $n = 2$ it is true).

If $d \leq 2n-3$, then we cut X by a generic hyperplane H. It turns out (using Bertini's theorem) that $X \cap H \subseteq \mathbb{P}^{n-1}$ is the generic smooth hypersurface of degree $d \leq 2(n-1)-1$, so by induction there exist a line $L \subseteq H$ and a point $P \in \mathbb{P}^n$ such that $(X \cap H) \cap L = \{P\}$. So we conclude that $X \in \mathcal{V}_d^{n-1}$.

It is known that the generic hypersurface of degree $d \geq 2n-2$ does not contain lines. So if $d = 2n-2$ or $d = 2n-1$, the statement follows by a). □

Corollary 3.8. *Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d, and let I be the ideal defining the Segre product $X \times \mathbb{P}^1 \subseteq \mathbb{P}^{2n+1}$. If $d \leq 2n-3$ or if $d \leq 2n-1$ and X is generic, then*

$$\mathrm{ara}_h(I) \leq 2n.$$

Proof. Just combine the above lemma with Theorem 3.4. □

Putting some stronger assumptions on the hypersurfaces we can even compute the arithmetical rank of the ideal defining their Segre product with $\mathbb{P}^1$ (and not just give an upper bound as in Theorem 3.4).

Theorem 3.9. *Let $X = \mathcal{V}_+(F) \subseteq \mathbb{P}^n$ be such that $F = x_n^d + \sum_{i=0}^{n-3} x_i G_i(x_0, \ldots, x_n)$ (G_i homogeneous polynomials of degree $d-1$), and let I be the ideal defining the Segre product $X \times \mathbb{P}^1 \subseteq \mathbb{P}^{2n+1}$. Then*

$$\mathrm{ara}_h(I) \leq 2n-1.$$

Moreover, if X is smooth, then

$$\mathrm{ara}_h(I) = 2n-1.$$

Furthermore, if k has characteristic 0, then $\mathrm{ara}(I) = \mathrm{ara}_h(I) = 2n-1$.

Proof. We can assume that k is algebraically closed. If X is smooth, by Proposition 2.16 écd$(\mathbb{P}^{2n+1} \setminus X) \geq 4n-1$, and equation 10 yields $\mathrm{ara}_h(I) \geq 2n-1$. If $\mathrm{char}(k) = 0$ Proposition 2.6 implies that $\mathrm{ara}(I) \geq 2n-1$.

Now we prove that the upper bound holds. Consider the set $\Omega = \{[i,j] : i < j, i+j \leq 2n-3\}$. As in the proof of Theorem 3.4, we have

$$\mathrm{ara}(\Omega R) \leq \mathrm{rank}(\Omega) = 2n-3.$$

Now the proof is completely analogous to that of Theorem 3.4. □

Remark 3.10. Notice that if $n \geq 4$, the generic hypersurface of $\mathbb{P}^n$ defined by the form $F = x_n^d + \sum_{i=0}^{n-3} x_i G_i(x_0, \ldots, x_n)$ is smooth (whereas if $n \leq 3$ and $d \geq 2$ such a hypersurface is always singular).

The below argument uses ideas from [SW]. Unfortunately, to use these kinds of tools we have to make some restrictions to $\mathrm{char}(k)$.

Theorem 3.11. *Assume* $\operatorname{char}(k) \neq 2$. *Let* $C = \mathcal{V}_+(F)$ *be a conic of* $\mathbb{P}^2$, *and let* I *be the ideal defining the Segre product* $X = C \times \mathbb{P}^m \subseteq \mathbb{P}^{3m+2}$. *Then*

$$\operatorname{ara}_h(I) = 3m.$$

In particular, X *is a set-theoretic complete intersection if and only if* $m = 1$. *Moreover, if* $\operatorname{char}(k) = 0$, *then* $\operatorname{ara}(I) = \operatorname{ara}_h(I) = 3m$.

Proof. First we want to give $3m$ homogeneous polynomials of $S = k[z_{ij} : i = 0, 1, 2, \quad j = 0, \dots, m]$ which define I up to radical.

For $i = 0, \dots, m$ choose F_i as in Remark 3.3. Then, for all $0 \leq j < i \leq m$, set

$$F_{ij} = \sum_{k=0}^{2} \frac{\partial F}{\partial x_k}(z_{0i}, z_{1i}, z_{2i}) z_{jk}.$$

Finally we set $G_h = \sum_{i+j=h} F_{ij}$ for all $h = 1, \dots, 2m-1$. We claim that

$$I = \sqrt{J}, \text{ where } J = (F_i, G_j : i = 0, \dots, m, \quad j = 1, \dots, 2m-1).$$

The inclusion $J \subseteq I$ follows from Euler's formula, since $\operatorname{char}(k) \neq 2$.

We can assume k algebraically closed by Remark 3.1, so we have to prove that $I \subseteq \sqrt{J}$, i.e., by the Nullstellensatz, that $\mathcal{Z}(J) \subseteq \mathcal{Z}(I)$. Pick $P \in \mathcal{Z}(J)$, and write P as $P = [P_0, P_1, \dots, P_m]$, where $P_j = [P_{0j}, P_{1j}, P_{2j}]$. Since $F_i(P) = 0$, for every $i = 0, \dots, m$ $P_i = 0$ or $P_i \in C$. So we have to prove that the P_i's that are non-zero are equal as points of $\mathbb{P}^2$.

By contradiction, let i be the minimum integer such that $P_i \neq 0$ and there exists k such that $P_k \neq 0$ and $P_i \neq P_k$ as points of $\mathbb{P}^2$, and set j the least among these k (so $i < j$). Set $h = i + j$. We claim that $P_k = P_l$ provided that $k + l = h$, $k < l$, $k \neq i$, $P_k \neq 0$ and $P_l \neq 0$.

In fact, if $l < j$, then $P_i = P_l$ by the choice of j. But for the same reason also $P_k = P_i$, so $P_k = P_l$. If $l > j$, then $k < i$, so it follows that $P_k = P_l$ by the choice of i. So $F_{lk}(P) = 0$, because P_k belongs to the tangent of C in P_l (being $P_l = P_k$). Then $G_h(P) = F_{ji}(P)$, and so, since $P \in \mathcal{Z}(J)$, $F_{ji}(P) = 0$. This means that P_i belongs to the tangent line of C in P_j, which is possible, C being a conic, only if $P_i = P_j$, a contradiction.

For the lower bound, we can assume that k is algebraically closed as in the proof of Corollary 3.5. By Proposition 2.16 $\operatorname{écd}(\mathbb{P}^{2n+1} \setminus X) \geq 4n - 1$, and equation (10) yields $\operatorname{ara}_h(I) \geq 2n - 1$. If $\operatorname{char}(k) = 0$, Proposition 2.6 implies that $\operatorname{ara}(I) \geq 2n - 1$. □

Remark 3.12. Bădescu and Valla recently computed in [BaVa], independently from this work, the arithmetical rank of the ideal defining any rational normal scroll. Since the Segre product of a conic with $\mathbb{P}^m$ is a rational normal scroll, Theorem 3.11 is a particular case of their result.

We end the paper with a proposition that yields a natural question.

Proposition 3.13. *Let* $X = \mathcal{V}_+(F) \subseteq \mathbb{P}^n$ *be a hypersurface smooth over a field of characteristic* 0 *and let* $I \subseteq P = k[z_0, \dots, z_N]$ *be the ideal defining* $Z = X \times \mathbb{P}^m \subseteq \mathbb{P}^N$ *(any embedding), with* $m \geq 1$. *Then*

$$\operatorname{cd}(P, I) = \begin{cases} N - 1 & \text{if } n = 2 \text{ and } \deg(F) \geq 3, \\ N - 2 & \text{otherwise.} \end{cases}$$

Proof. By Remark 2.2 we can assume $k = \mathbb{C}$. Using equation (5) we have

$$\beta_0(Z) = 1, \quad \beta_1(Z) = \beta_1(X) \quad \text{and} \quad \beta_2(Z) = \beta_2(X) + 1 \geq 2.$$

If $n = 2$, notice that $\beta_1(X) \neq 0$ if and only if $\deg(F) \geq 3$. In fact, by equation (6),

$$\beta_1(X) = h^{01}(X) + h^{10}(X) = 2h^{01}(X)$$

(the last equality comes from Serre's duality). But $h^{01}(X)$ is the geometric genus of X, therefore it is different from 0 if and only if $\deg(F) \geq 3$. So if $n = 2$ we conclude by Theorem 2.4.

If $n > 2$ we have $\beta_1(X) = 0$ by the Lefschetz hyperplane theorem [La, Theorem 3.1.17]; therefore we conclude by Theorem 2.4. □

In light of the above proposition, the following question is natural.

Question 3.14. With the notation of Proposition 3.13, if we consider the Segre embedding of Z (and so $N = nm + n + m$), do the integers $\operatorname{ara}(I)$ and $\operatorname{ara}_h(I)$ depend only on n, m and $\deg(F)$?

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