Singularities, Serre conditions and h-vectors

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Notation

Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a standard graded algebra over a field $R_0 = K$. The *Hilbert series* of R is

$$H_R(t) = \sum_{i \in \mathbb{N}} (\dim_K R_i) t^i \in \mathbb{N}[[t]].$$

If $d = \dim R$, Hilbert proved that

$$H_R(t) = \frac{h(t)}{(1-t)^d}$$

where $h(t) = h_0 + h_1 t + h_2 t^2 + \ldots + h_s t^s \in \mathbb{Z}[t]$ is the *h-polynomial* of R. We will name the coefficients vector $(h_0, h_1, h_2, \ldots, h_s)$ the *h-vector* of R.

Notation

Let $X = \operatorname{Proj} R$. If $\dim_K R_1 = n+1$, R is the coordinate ring of the embedding $X \subset \mathbb{P}^n$, whose *degree* is $h(1) = \sum_{i \geq 0} h_i$. So the sum of the h_i is positive, but it can happen that some of the h_i is negative.

If R is Cohen-Macaulay, then it is easy to see that $h_i \geq 0$ for all $i \geq 0$, however without the CM assumption things get complicated. In this talk I want to discuss conditions on R and/or on X which ensure at least the nonnegativity of the first h_i's and that the degree of $X \subset \mathbb{P}^n$ is bounded below by their sum.

Serre conditions

For $r \in \mathbb{N}$, we say that R *satisfies the Serre condition* (S_r) if:

$$\operatorname{depth} R_{\mathfrak{p}} \geq \min\{\dim R_{\mathfrak{p}}, r\} \quad \forall\ \mathfrak{p} \in \operatorname{Spec} R.$$

It turns out that this is equivalent to $\operatorname{depth} R \geq \min\{\dim R, r\}$ and

$$\operatorname{depth} \mathcal{O}_{X,x} \geq \min\{\dim \mathcal{O}_{X,x}, r\} \quad \forall\ x \in X.$$

In particular, if X is nonsingular, R satisfies the Serre condition (S_r) if and only if $\operatorname{depth} R \geq \min\{\dim R, r\}$.

Serre conditions

Notice that R is Cohen-Macaulay if and only if R satisfies condition (S_i) for all $i \in \mathbb{N}$. Since if R is CM $h_i \geq 0$ for all $i \in \mathbb{N}$, it is natural to ask:

Question

If R satisfies (S_r), is it true that $h_i \geq 0$ for all $i = 0, \ldots, r$?

The above question is known to have a positive answer for Stanley-Reisner rings R (i.e. if R is defined by squarefree monomial ideals) by a result of Murai and Terai ...

Serre conditions

It is too optimistic to expect a positive answer to the previous question in general though: it is not very difficult to construct, for all $r \geq 2$, an $(r+1)$-dimensional R satisfying (S_r) with h-vector

$$(1, r+1, -1).$$

Such an R can be even chosen to be Buchsbaum of Castelnuovo-Mumford regularity $\operatorname{reg} R = 1$. So the question must be adjusted:

Question

If R satisfies (S_r) and X has nice singularities, is it true that $h_i \geq 0$ for all $i = 0, \ldots, r$?

The main result

Theorem (Dao-Ma-_)

Let R satisfy Serre condition (S_r). Suppose either

- K has characteristic 0 and X is Du Bois, or
- K has positive characteristic and X is globally F-split.

Then $h_i \geq 0$ for all $i = 0, \ldots, r$ and the degree of $X \subset \mathbb{P}^n$ is at least $h_0 + h_1 + \ldots + h_{r-1}$. Furthermore, if $\operatorname{reg} R < r$, or if $h_i = 0$ for some $1 \leq i \leq r$, then R is Cohen-Macaulay.

The characteristic 0 version of the previous result is much stronger than the positive characteristic one: in fact, we have the following:

- If $\text{char}(K) = 0$, R Stanley-Reisner ring $\Rightarrow X$ Du Bois.
- If $\text{char}(K) = 0$, X nonsingular $\Rightarrow X$ Du Bois.
- If $\text{char}(K) > 0$, R S-R ring $\Rightarrow X$ globally F-split.
- If $\text{char}(K) > 0$, X nonsingular $\not\Rightarrow X$ globally F-split.

The definition of Du Bois variety (over a field of characteristic 0) is quite involved, it is worth, however, to notice the following:

We say that X is *locally Stanley-Reisner* if $\widehat{\mathcal{O}_{X,x}} \cong K[[\Delta_x]]$ for all $x \in X$ (for some simplicial complex Δ_x). We have:

- X locally Stanley-Reisner $\Rightarrow$ X Du Bois.
- R Stanley-Reisner ring $\Rightarrow$ X locally Stanley-Reisner.
- X nonsingular $\Rightarrow$ X locally Stanley-Reisner.

The condition MT_r

Let M be a finitely generated graded S-module, where $S = K[X_0, \ldots, X_n]$. We say that M satisfies the condition MT_r if

$$\operatorname{reg} \operatorname{Ext}_S^{n+1-i}(M, \omega_S) \leq i - r \quad \forall\, i = 0, \ldots, \dim M - 1.$$

This notion is good for several reasons:

- The condition MT_r does not depend on S.
- The condition MT_r is preserved by taking general hyperplane sections.
- The condition MT_r is preserved by saturating.

Lemma (Murai-Terai, Dao-Ma-_)

Let M be a finitely generated graded S-module generated in degree ≥ 0 with h-vector $(h_0, \ldots, h_s)$ satisfying MT_r. Then

- $h_i \geq 0$ for all $i \leq r$.
- $h_r + h_{r+1} + \ldots + h_s \geq 0$, or equivalently the multiplicity of M is at least $h_0 + h_1 + \ldots + h_{r-1}$.

Furthermore, if $\operatorname{reg} M < r$ or M is generated in degree 0 and $h_i = 0$ for some $i \leq r$, then M is Cohen-Macaulay.

The condition MT_r

Let $\dim_K R_1 = n+1$ and $S = K[X_0, \ldots, X_n]$. We want to show that, if R satisfies Serre condition (S_r), then it also satisfies MT_r, namely

$$\operatorname{reg} \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \leq i - r \quad \forall\ i = 0, \ldots, \dim R - 1,$$

provided X is Du Bois (in characteristic 0) or globally F-split (in positive characteristic). By the previous lemma this would imply the desired result.

Dimensions of Ext's

It is simple to show that

$$\dim \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \le i \quad \forall\ i = 0, \ldots, \dim R - 1,$$

with equality holding iff $\dim R/\mathfrak{p} = i$ for some associated prime $\mathfrak{p}$ of R. A similar argument, plus the fact that R is unmixed as soon as it satisfies (S_2), shows that the following are equivalent for any natural number $r \ge 2$:

1. R satisfies Serre condition (S_r).
2. $\dim \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \le i - r \quad \forall\ i \in \mathbb{N}$.

Regularity of Ext's

So, under our assumptions on X, in order to prove that R satisfies MT_r provided it satisfies (S_r), it is enough to show that

$$\operatorname{reg} \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \leq \dim \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \quad \forall\, i = 0, \ldots, \dim R - 1.$$

We show more:

Dao-Ma-_

Let $\mathfrak{m}$ be the homogeneous maximal ideal of S and assume either

- K has characteristic 0 and X is Du Bois, or
- K has positive characteristic and X is globally F-split.

Then $H^j_{\mathfrak{m}}(\operatorname{Ext}^i_S(R, \omega_S))_{>0} = 0$ for all $i, j \in \mathbb{N}$. In particular, $\operatorname{reg} \operatorname{Ext}^i_S(R, \omega_S) \leq \dim \operatorname{Ext}^i_S(R, \omega_S)$ for all $i \in \mathbb{N}$.

A corollary

Corollary

Let $R = S/I$ satisfies (S_r) and assume I has height c and does not contain elements of degree $< r$. Suppose either

- K has characteristic 0 and X is Du Bois, or
- K has positive characteristic and X is globally F-split.

Then, the minimal generators of I of degree r are $\leq \binom{c+r-1}{r}$; if equality holds, then R is Cohen-Macaulay.

If $r = 2$, the above corollary is true just assuming that X is reduced...

Questions:

Is it true that $h_i \geq 0$ for all $i \leq r$ provided R satisfies (S_r) and either

1. K has characteristic 0 and X is Du Bois in codimension $r-2$, or
2. K has positive characteristic and X has F-injective singularities ???

THANK YOU !

Some references:

1. H. Dao, L. Ma, M. Varbaro *Regularity, singularities and h-vector of graded algebras*, arXiv:1901.01116.
2. S. Murai, N. Terai, *h-vectors of simplicial complexes with Serre's conditions*, Math. Res. Lett. 16 (2009), no. 6, 1015-1028.