F-THRESHOLDS, INTEGRAL CLOSURE AND CONVEXITY

MATTEO VARBARO

To Winfried Bruns on his 70th birthday

ABSTRACT. The purpose of this note is to revisit the results of [HV] from a slightly different perspective, outlining how, if the integral closures of a finite set of prime ideals abide the expected convexity patterns, then the existence of a peculiar polynomial f allows one to compute the F-jumping numbers of all the ideals formed by taking sums of products of the original ones. The note concludes with the suggestion of a possible source of examples falling in such a framework.

1. PROPERTIES A, A+ AND B FOR A FINITE SET OF PRIME IDEALS

Let S be a standard graded polynomial ring over a field $\mathbb{k}$ and let m be a positive integer. Fix homogeneous prime ideals of S:

$$\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m.$$

For any $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{N}^m$ and $k = 1, \ldots, m$, denote by

$$I^\sigma := \mathfrak{p}_1^{\sigma_1} \cdots \mathfrak{p}_m^{\sigma_m} \quad \text{and} \quad e_k(\sigma) := \max\{\ell : I^\sigma \subseteq \mathfrak{p}_k^{(\ell)}\}.$$

Obviously we have $I^\sigma \subseteq \bigcap_{k=1}^m \mathfrak{p}_k^{(e_k(\sigma))}$. Since $S_{\mathfrak{p}_k}$ is a regular local ring with maximal ideal $(\mathfrak{p}_k)_{\mathfrak{p}_k}$, we have that $(\mathfrak{p}_k)_{\mathfrak{p}_k}^\ell$ is integrally closed in $S_{\mathfrak{p}_k}$ for any $\ell \in \mathbb{N}$. Therefore $p_k^{(\ell)} = (\mathfrak{p}_k)_{\mathfrak{p}_k}^\ell \cap S$ is integrally closed in S for any $\ell \in \mathbb{N}$. Eventually we conclude that $\bigcap_{k=1}^m \mathfrak{p}_k^{(e_k(\sigma))}$ is integrally closed in S, so:

$$\overline{I^\sigma} \subseteq \bigcap_{k=1}^m \mathfrak{p}_k^{(e_k(\sigma))}. \tag{1}$$

Definition 1.1. We say that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy condition **A** if

$$\overline{I^\sigma} = \bigcap_{k=1}^m \mathfrak{p}_k^{(e_k(\sigma))} \quad \forall\, \sigma \in \mathbb{N}^m.$$

If $\Sigma \subseteq \mathbb{N}^m$, denote by $I(\Sigma) := \sum_{\sigma \in \Sigma} I^\sigma$ and by $\overline{\Sigma} \subseteq \mathbb{Q}^m$ the convex hull of $\Sigma \subseteq \mathbb{Q}^m$.

Lemma 1.2. *For any $\Sigma \subseteq \mathbb{N}^m$, $\overline{I(\Sigma)} \supseteq \sum_{\mathbf{v} \in \overline{\Sigma}} I^{\lceil \mathbf{v} \rceil}$, where $\lceil \mathbf{v} \rceil := (\lceil v_1 \rceil, \ldots, \lceil v_m \rceil)$ for $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbb{Q}^m$.*

Proof. Since S is Noetherian, we can assume that $\Sigma = \{\sigma^1, \ldots, \sigma^N\}$ is a finite set. Take $\mathbf{v} \in \overline{\Sigma}$. Then there exist nonnegative rational numbers $q_1, \ldots, q_N$ such that

$$\mathbf{v} = \sum_{i=1}^N q_i \sigma^i \quad \text{and} \quad \sum_{i=1}^N q_i = 1.$$

Let d be the product of the denominators of the q_i's and $\sigma = d \cdot \mathbf{v} \in \mathbb{N}^m$. Clearly:

$$(I^{\lceil \mathbf{v} \rceil})^d = I^{d \cdot \lceil \mathbf{v} \rceil} \subseteq I^{\sigma}.$$

Setting $a_i = dq_i$, notice that $\sigma = \sum_{i=1}^N a_i \sigma^i$ and $\sum_{i=1}^N a_i = d$. Therefore

$$(I^{\lceil \mathbf{v} \rceil})^d \subseteq I^{\sigma} \subseteq I(\Sigma)^d.$$

This implies that $I^{\lceil \mathbf{v} \rceil}$ is contained in the integral closure of $I(\Sigma)$. □

From the above lemma, $\overline{I(\Sigma)} \supseteq \sum_{\mathbf{v} \in \overline{\Sigma}} \overline{I^{\lceil \mathbf{v} \rceil}}$. In particular:

$$\mathfrak{p}_1, \ldots, \mathfrak{p}_m \text{ satisfy condition } \mathbf{A} \implies \overline{I(\Sigma)} \supseteq \sum_{\mathbf{v} \in \overline{\Sigma}} \left(\bigcap_{k=1}^m \mathfrak{p}_k^{(e_k(\lceil \mathbf{v} \rceil))} \right). \tag{2}$$

Definition 1.3. We say that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy condition **A+** if

$$\overline{I(\Sigma)} = \sum_{\mathbf{v} \in \overline{\Sigma}} \left(\bigcap_{k=1}^m \mathfrak{p}_k^{(e_k(\lceil \mathbf{v} \rceil))} \right) \quad \forall\, \Sigma \subseteq \mathbb{N}^m$$

Remark 1.4. If $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy condition **A+**, then they satisfy **A** as well (for $\sigma \in \mathbb{N}^m$, just consider the singleton $\Sigma = \{\sigma\}$).

Lemma 1.5. *Let $\sigma^1, \ldots, \sigma^N$ be vectors in $\mathbb{N}^m$, and $a_1, \ldots, a_N \in \mathbb{N}$. Then*

$$e_k\left(\sum_{i=1}^N a_i \sigma^i\right) = \sum_{i=1}^N a_i e_k(\sigma^i) \quad \forall\, k = 1, \ldots, m.$$

Proof. Set $\sigma = \sum_{i=1}^N a_i \sigma^i$, and notice that

$$I^{\sigma} = \prod_{i=1}^N \left(I^{\sigma^i}\right)^{a_i} \subseteq \prod_{i=1}^N \left(\mathfrak{p}_k^{(e_k(\sigma^i))}\right)^{a_i} \subseteq \prod_{i=1}^N \mathfrak{p}_k^{(a_i e_k(\sigma^i))} \subseteq \mathfrak{p}_k^{(\sum_{i=1}^N a_i e_k(\sigma^i))},$$

so the inequality $e_k(\sigma) \geq \sum_{i=1}^N a_i e_k(\sigma^i)$ follows directly from the definition.

For the other inequality, for each $i = 1, \ldots, N$ choose $f_i \in I^{\sigma^i}$ such that its image in $S_{\mathfrak{p}_k}$ is not in $(\mathfrak{p}_k)_{\mathfrak{p}_k}^{e_k(\sigma^i)+1}$. Then the class $\overline{f_i}$ is a nonzero element of degree $e_k(\sigma^i)$ in the associated graded ring G of $S_{\mathfrak{p}_k}$. Since G is a polynomial ring (in particular a domain), the element $\prod_{i=1}^N \overline{f_i}^{a_i}$ is a nonzero element of degree $\sum_{i=1}^N a_i e_k(\sigma^i)$ in G. Therefore

$$\prod_{i=1}^N f_i^{a_i} \in I^{\sigma}_{\mathfrak{p}_k} \setminus \mathfrak{p}_k^{(\sum_{i=1}^N a_i e_k(\sigma^i)+1)}.$$

This means that $e_k(\sigma) \leq \sum_{i=1}^N a_i e_k(\sigma^i)$. □

Consider the function $e : \mathbb{N}^m \to \mathbb{N}^m$ defined by

$$\sigma \mapsto e(\sigma) := (e_1(\sigma), \ldots, e_m(\sigma)).$$

From the above lemma we can extend it to a $\mathbb{Q}$-linear map $e : \mathbb{Q}^m \to \mathbb{Q}^m$.

Given $\Sigma \subseteq \mathbb{N}^m$, the above map sends $\overline{\Sigma}$ to the convex hull $P_{\Sigma} \subseteq \mathbb{Q}^m$ of the set $\{e(\sigma) : \sigma \in \Sigma\} \subseteq \mathbb{Q}^m$. In particular we have the following:

Proposition 1.6. *The prime ideals* $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ *satisfy condition* **A+** *if and only if*

$$\overline{I(\Sigma)} = \sum_{(v_1,\ldots,v_m)\in P_\Sigma} \left(\bigcap_{k=1}^{m} \mathfrak{p}_k^{(\lceil v_k \rceil)} \right) \quad \forall\, \Sigma \subseteq \mathbb{N}^m$$

If $\Sigma \subseteq \mathbb{N}^m$ and $s \in \mathbb{N}$, define $\Sigma^s := \{\sigma^{i_1} + \ldots + \sigma^{i_s} : \sigma^{i_k} \in \Sigma\}$. Then

$$I(\Sigma)^s = I(\Sigma^s).$$

Furthermore $\overline{\Sigma^s} = s \cdot \overline{\Sigma}$, i.e. $P_{\Sigma^s} = s \cdot P_\Sigma$. So:

Proposition 1.7. *If* $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ *satisfy condition* **A+**, *then*

$$\overline{I(\Sigma)^s} = \sum_{(v_1,\ldots,v_m)\in P_\Sigma} \left(\bigcap_{k=1}^{m} \mathfrak{p}_k^{(\lceil s v_k \rceil)} \right) \quad \forall\, \Sigma \subseteq \mathbb{N}^m,\ s \in \mathbb{N}.$$

We conclude this section by stating the following definition:

Definition 1.8. We say that $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ satisfy condition **B** if there exists a polynomial $f \in \bigcap_{k=1}^m \mathfrak{p}_k^{\mathrm{ht}(\mathfrak{p}_k)}$ such that $\mathrm{in}_\prec(f)$ is a square-free monomial for some term order $\prec$ on S.

Example 1.9. Let $S = \mathbb{k}[x,y]$, $m = 2$ and $\mathfrak{p}_1 = (x)$ and $\mathfrak{p}_2 = (y)$. Of course these ideals satisfy condition **B** by considering $f = xy$.

If $\sigma = (\sigma_1, \sigma_2)$, then $I^\sigma = (x^{\sigma_1} y^{\sigma_2})$ and $e_k(\sigma) = \sigma_k$, therefore they trivially satisfy condition **A** so.

Though less trivial, it is a well-known fact that (x) and (y) satisfy condition **A+** as well: for example, $\overline{(x^3, y^3)} = (x^3, x^2y, xy^2, y^3)$.

2. Generalized test ideals and F-thresholds

Let $p > 0$ be the characteristic of $\mathbb{k}$, I be an ideal of S and $\mathfrak{m}$ be the homogeneous maximal ideal of S. For all $e \in \mathbb{N}$, denoting by $\mathfrak{m}^{[p^e]} = (x_1^{p^e}, \ldots, x_n^{p^e})$, define

$$\nu_e(I) := \max\{r \in \mathbb{N} : I^r \not\subseteq \mathfrak{m}^{[p^e]}\}.$$

The *F-pure threshold* of I is then

$$\mathrm{fpt}(I) := \lim_{e\to\infty} \frac{\nu_e(I)}{p^e}.$$

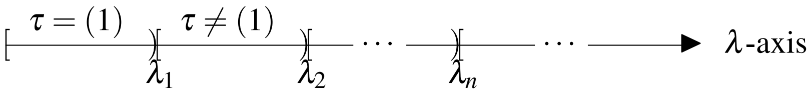
The *p^e-th root of I*, denoted by $I^{[1/p^e]}$, is the smallest ideal $J \subseteq S$ such that $I \subseteq J^{[p^e]}$. By the flatness of the Frobenius over S the q-th root is well defined. If λ is a positive real number, then it is easy to see that

$$\left(I^{\lceil \lambda p^e \rceil} \right)^{[1/p^e]} \subseteq \left(I^{\lceil \lambda p^{e+1} \rceil} \right)^{[1/p^{e+1}]}.$$

The *generalized test ideal* of I with coefficient λ is defined as:

$$\tau(\lambda \cdot I) \underset{e \gg 0}{:=} \left(I^{\lceil \lambda p^e \rceil} \right)^{[1/p^e]}.$$

Note that $\tau(\lambda \cdot I) \supseteq \tau(\mu \cdot I)$ whenever $\lambda \leq \mu$. By [BMS, Corollary 2.16], $\forall\ \lambda \in \mathbb{R}_{>0}$, $\exists\ \varepsilon \in \mathbb{R}_{>0}$ such that $\tau(\lambda \cdot I) = \tau(\mu \cdot I)\ \ \forall\ \mu \in [\lambda, \lambda+\varepsilon)$. A $\lambda \in \mathbb{R}_{>0}$ is called an *F-jumping number* for I if $\tau((\lambda-\varepsilon)\cdot I) \supsetneq \tau(\lambda \cdot I)\ \ \forall\ \varepsilon \in \mathbb{R}_{>0}$.



$$(1) \supsetneq \tau(\lambda_1 \cdot I) \supsetneq \tau(\lambda_2 \cdot I) \supsetneq \ldots \supsetneq \tau(\lambda_n \cdot I) \supsetneq \ldots$$

The λ_i above are the F-jumping numbers. Notice that $\lambda_1 = \operatorname{fpt}(I)$.

Theorem 2.1. *If $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy conditions* **A** *and* **B**, *then $\forall\ \lambda \in \mathbb{R}_{>0}$ we have*

$$\tau(\lambda \cdot I^{\sigma}) = \bigcap_{k=1}^{m} \mathfrak{p}_k^{(\lfloor \lambda e_k(\sigma) \rfloor + 1 - \operatorname{ht}(\mathfrak{p}_k))} \qquad \forall\ \sigma \in \mathbb{N}^m.$$

If $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy conditions **A+** *and* **B**, *then $\forall\ \lambda \in \mathbb{R}_{>0}$ we have*

$$\tau(\lambda \cdot I(\Sigma)) = \sum_{(v_1,\ldots,v_m)\in P_{\Sigma}} \left(\bigcap_{k=1}^{m} \mathfrak{p}_k^{(\lfloor \lambda v_k \rfloor + 1 - \operatorname{ht}(\mathfrak{p}_k))} \right) \qquad \forall\ \Sigma \subseteq \mathbb{N}^m.$$

Proof. The first part immediately follows from [HV, Theorem 3.14], for if $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy conditions **A** and **B**, then I^{σ} obviously enjoys condition $(\diamond+)$ of [HV] $\forall\ \sigma \in \mathbb{N}^m$.

Concerning the second part, Proposition 1.7 implies that $I(\Sigma)$ enjoys condition $(*)$ of [HV] $\forall\ \Sigma \subseteq \mathbb{N}^m$ whenever $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ satisfy conditions **A+** and **B**. Therefore the conclusion follows once again by [HV, Theorem 4.3]. ☐

3. WHERE TO FISH?

Let $\Bbbk$ be of characteristic $p > 0$. So far we have seen that, if we have graded primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of S enjoying **A** and **B**, then we can compute lots of generalized test ideals. If they enjoy **A+** and **B**, we get even more.

That looks nice, but how can we produce $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ like these? Before trying to answer this question, let us notice that, as explained in [HV], the ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of the following examples satisfy conditions **A+** and **B**:

(i) $S = \Bbbk[x_1, \ldots, x_m]$ and $\mathfrak{p}_k = (x_k)$ for all $k = 1, \ldots, m$.
(ii) $S = \Bbbk[X]$, where X is an $m \times n$ generic matrix (with $m \leq n$) and $\mathfrak{p}_k = I_k(X)$ is the ideal generated by the k-minors of X for all $k = 1, \ldots, m$.
(iii) $S = \Bbbk[Y]$, where Y is an $m \times m$ generic symmetric matrix and $\mathfrak{p}_k = I_k(Y)$ is the ideal generated by the k-minors of Y for all $k = 1, \ldots, m$.
(iv) $S = \Bbbk[Z]$, where Z is a $(2m+1) \times (2m+1)$ generic skew-symmetric matrix and $\mathfrak{p}_k = P_{2k}(Z)$ is the ideal generated by the $2k$-Pfaffians of Z for all $k = 1, \ldots, m$.

Even for a simple example like (i), Theorem 2.1 is interesting: it gives a description of the generalized test ideals of any monomial ideal.

In my opinion, a class to look at to find new examples might be the following: **fix** $f \in S$ **a homogeneous polynomial such that** $\operatorname{in}_{\prec}(f)$ **is a square-free monomial for**

some term order $\prec$ (better if lexicographical) **on** S, and let $\mathscr{C}_f$ be the set of ideals of S defined, recursively, like follows:

(a) $(f) \in \mathscr{C}_f$;
(b) If $I \in \mathscr{C}_f$, then $I : J \in \mathscr{C}_f$ for all $J \subseteq S$;
(c) If $I, J \in \mathscr{C}_f$, then both $I+J$ and $I \cap J$ belong to $\mathscr{C}_f$.

If f is an irreducible polynomial, $\mathscr{C}_f$ consists of only the principal ideal generated by f, but otherwise things can get interesting. Let us give two guiding examples:

(i) If $u := x_1 \cdots x_m$, then the associated primes of (u) are $(x_1), \ldots, (x_m)$. Furthermore all the ideals of $S = \Bbbk[x_1, \ldots, x_m]$ generated by variables are sums of the principal ideals above, and all square-free monomial ideals can be obtained by intersecting ideals generated by variables. Therefore, any square-free monomial ideal belongs to $\mathscr{C}_u$, and one can check that indeed:

$$\mathscr{C}_u = \{\text{square-free monomial ideals of } S\}.$$

(ii) Let $X = (x_{ij})$ be an $m \times n$ matrix of variables, with $m \leq n$. For positive integers $a_1 < \ldots < a_k \leq m$ and $b_1 < \ldots < b_k \leq n$, recall the standard notation for the corresponding k-minor:

$$[a_1, \ldots, a_k | b_1, \ldots, b_k] := \det \begin{pmatrix} x_{a_1 b_1} & x_{a_1 b_2} & \cdots & x_{a_1 b_k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{a_k b_1} & x_{a_k b_2} & \cdots & x_{a_k b_k} \end{pmatrix}.$$

For $i = 0, \ldots, n-m$, let $\delta_i := [1, \ldots, m | i+1, \ldots, m+i]$. Also, for $j = 1, \ldots, m-1$ set $g_j := [j+1, \ldots, m | 1, \ldots, m-j]$ and $h_j := [1, \ldots, m-j | n-m+j+1, \ldots, n]$.

Let Δ be the product of the δ_i's, the g_j's and the h_j's:

$$\Delta := \prod_{i=0}^{n-m} \delta_i \cdot \prod_{j=1}^{m-1} g_j h_j.$$

By considering the lexicographical term order $\prec$ extending the linear order

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{2n} > \cdots > x_{m1} > \cdots > x_{mn},$$

we have that

$$\operatorname{in}(\Delta) = \prod_{i=0}^{n-m} \operatorname{in}(\delta_i) \cdot \prod_{j=1}^{m-1} \operatorname{in}(g_j) \operatorname{in}(h_j) = \prod_{\substack{i \in \{1,\ldots,m\} \\ j \in \{1,\ldots,n\}}} x_{ij}$$

is a square-free monomial. Since each (δ_i) belongs to $\mathscr{C}_\Delta$, the height-$(n-m+1)$ complete intersection

$$J := (\delta_0, \ldots, \delta_{n-m})$$

is an ideal of $\mathscr{C}_\Delta$ too. Notice that the ideal $I_m(X)$ generated by all the maximal minors of X is a height-$(n-m+1)$ prime ideal containing J. So $I_m(X)$ is an associated prime of J, and thus an ideal of $\mathscr{C}_\Delta$ by definition. With more effort, one should be able to show that the ideals of minors $I_k(X)$ stay in $\mathscr{C}_\Delta$ for any size k.

The ideals of $\mathscr{C}_f$ have quite strong properties. First of all, $\mathscr{C}_f$ is a finite set by [Sc]. Then, all the ideals in $\mathscr{C}_f$ are radical. Even more, Knutson proved in [Kn] that they have a square-free initial ideal!

In order to produce graded prime ideals $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ satisfying conditions **A** (or even **A+**) and **B**, it seems natural to seek for them among the prime ideals in $\mathscr{C}_f$. This is because, at least, f is a good candidate for the polynomial needed for condition **B**: if $f = f_1\cdots f_r$ is the factorization of f in irreducible polynomials, then for each $A\subseteq\{1,\ldots,r\}$ the ideal

$$J_A := (f_i : i\in A)\subseteq S$$

is a complete intersection of height $|A|$. If $\mathfrak{p}$ is an associated prime ideal of J_A, then f obviously belongs to $\mathfrak{p}^{|A|}\subseteq\mathfrak{p}^{(|A|)}$. So such a $\mathfrak{p}$ satisfies **B**.

Question 3.1. *Does the ideal $\mathfrak{p}$ above satisfy condition* **A**? *Even more, is it true that for prime ideals $\mathfrak{p}$ as above $\mathfrak{p}^s=\mathfrak{p}^{(s)}$ for all $s\in\mathbb{N}$?*

If the above question admitted a positive answer, Theorem 2.1 would provide the generalized test ideals of $\mathfrak{p}$. A typical example, is when $J_A=(\delta_0,\ldots,\delta_{n-m})$ and $\mathfrak{p}=I_m(X)$ (see (ii) above), in which case it is well-known that $I_m(X)^s=I_m(X)^{(s)}$ for all $s\in\mathbb{N}$ (e.g. see [BV, Corollary 9.18].

Remark 3.2. Unfortunately, it is not true that $\mathfrak{p}$ satisfies **B** for all prime ideal $\mathfrak{p}\in\mathscr{C}_f$: for example, consider $f=\Delta$ in the case $m=n=2$, that is $\Delta = x_{21}(x_{11}x_{22}-x_{12}x_{21})x_{21}$. Notice that $(x_{21},x_{11}x_{22}-x_{12}x_{21})=(x_{21},x_{11}x_{22})=(x_{21},x_{11})\cap(x_{21},x_{22})$, so

$$\mathfrak{p}=(x_{21},x_{11})+(x_{21},x_{22})=(x_{21},x_{11},x_{22})\in\mathscr{C}_\Delta.$$

However $\Delta\notin\mathfrak{p}^{(3)}$.

Problem 3.3. *Find a large class of prime ideals in $\mathscr{C}_f$ (or even characterize them) satisfying condition* **B**.

If $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ are prime ideals satisfying **A+**, then (by definition)

$$\overline{\sum_{i\in A}\mathfrak{p}_i}=\sum_{i\in A}\mathfrak{p}_i \quad \forall\, A\subseteq\{1,\ldots,m\}.$$

If $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ are in $\mathscr{C}_f$, then the above equality holds true because $\sum_{i\in A}\mathfrak{p}_i$, belonging to $\mathscr{C}_f$, is a radical ideal.

Problem 3.4. *Let $\mathscr{P}_f$ be the set of prime ideals in $\mathscr{C}_f$. Is it true that $\mathscr{P}_f$ satisfies condition* **A+**? *If not, find a large subset of $\mathscr{P}_f$ satisfying condition* **A+**.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI GENOVA, ITALY
E-mail address: varbaro@dima.unige.it