Regularity of line arrangements

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Matteo Varbaro

Università degli Studi di Genova

Notions from algebraic geometry

Let X be a (possibly reducible) projective scheme over an algebraically closed field $\mathbb{k}$. We say that an embedding $X \subseteq \mathbb{P}^n$ is **arithmetically Gorenstein** if, denoting by $\mathcal{I}_X$ its sheaf of ideals:

- $X \subseteq \mathbb{P}^n$ is projectively normal, i. e. $H^1(X, \mathcal{I}_X(k)) = 0 \ \forall \ k \in \mathbb{Z}$;
- $H^i(X, \mathcal{O}_X(k)) = 0 \ \forall \ 0 < i < \dim X, \ \ k \in \mathbb{Z}$;
- $\omega_X \cong \mathcal{O}_X(a)$ for some $a \in \mathbb{Z}$.

In this setting the **Castelnuovo-Mumford regularity** of $X \subseteq \mathbb{P}^n$ is:

$$\operatorname{reg} X = \dim X + a + 2.$$

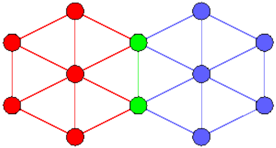
Examples

(a) If $X \subseteq \mathbb{P}^n$ is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_c$, where c is the codimension of X in $\mathbb{P}^n$, then the embedding is aGorenstein of regularity $d_1 + \ldots + d_c - c + 1$.
(b) If $X \subseteq \mathbb{P}^n$ is a projectively normal embedding of a Calabi-Yau manifold over $\mathbb{C}$, then it is aGorenstein of regularity $\dim X + 2$.

Notions from graph theory

Given a simple graph G on s vertices and an integer r less than s, we say that G is r-**connected** if the removal of less than r vertices of G does not disconnect it. The **valency** of a vertex v of G is:

$$\delta(v) = |\{w : \{v, w\} \text{ is an edge of } G\}|.$$



- 2-connected, not 3-connected.
- $\delta(\bullet) = 5$.
- $\delta(\text{inner}) = \delta(\text{inner}) = 6$.
- $\delta(\text{boundary}) = \delta(\text{boundary}) = 3$.

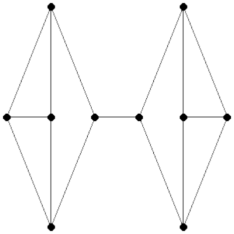
Remark

(i) G is 1-connected $\Leftrightarrow$ G is connected.

(ii) G is r-connected $\Rightarrow$ G is r'-connected for all $r' \leq r$.

(iii) G is r-connected $\Rightarrow$ $\delta(v) \geq r$ for all vertices v of G.

G is said to be r-**regular** if each vertex has valency r.



3-regular, connected, not 2-connected.

Let $X = \bigcup_{i=1}^{s} X_i \subseteq \mathbb{P}^n$ be a **line arrangement** (reduced union of lines). The **dual graph** of X is the simple graph $G(X)$ on s vertices where, for vertices $i \neq j$ in $\{1, \ldots, s\}$:

$$\{i,j\} \text{ is an edge of } G(X) \Leftrightarrow X_i \cap X_j \neq \emptyset$$

Example

If $X \subseteq \mathbb{P}^2$ then $G(X)$ is the complete graph on s vertices, K_s.

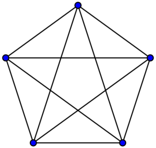


Figure : K_5

In this case X is a hypersurface:

- $\operatorname{reg} X - 1 = s - 1$.
- $G(X)$ is $(s-1)$-connected.
- $G(X)$ is $(s-1)$-regular.

Dual graphs

Example

If $Q \subseteq \mathbb{P}^3$ is a smooth quadric, and X is the union of p lines of a ruling of Q, and q of the other ruling, then $G(X)$ is the complete bipartite graph $K_{p,q}$.

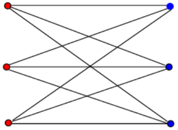


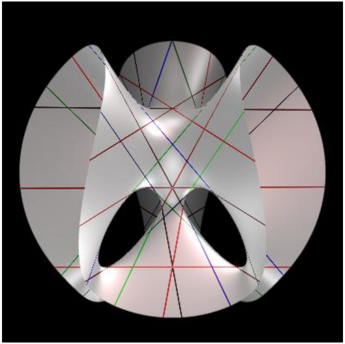
Figure : $K_{3,3}$

One can check that $X \subseteq \mathbb{P}^3$ is a complete intersection (of Q and an union of p planes) if and only if $p = q$: in this case

- $\operatorname{reg} X - 1 = p$.
- $G(X)$ is p-connected.
- $G(X)$ is p-regular.

Let $Z \subseteq \mathbb{P}^3$ be a smooth cubic, and $X = \bigcup_{i=1}^{27} X_i$ be the union of all the lines on Z. Below is a representation of the Clebsch's cubic:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = (x_0 + x_1 + x_2 + x_3)^3.$$

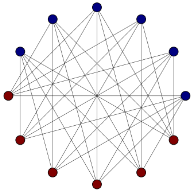


One can realize that $X \subseteq \mathbb{P}^3$ is a complete intersection of the cubic Z and a union of 9 planes. One can check that:

- $\operatorname{reg} X - 1 = 10$.
- $G(X)$ is 10-connected.
- $G(X)$ is 10-regular.

Schläfli double six

If, among the 27 lines on a smooth cubic, we take only the 6 corresponding to the exceptional divisors and the 6 corresponding to the strict transforms of the conics, we get a line arrangement $X \subseteq \mathbb{P}^3$ known as **Schläfli double six**. One can check that it is a complete intersection of the cubic and of a quartic; $G(X)$ is:



- $\operatorname{reg} X - 1 = 5$.
- $G(X)$ is 5-connected.
- $G(X)$ is 5-regular.

Theorem

Let $X \subseteq \mathbb{P}^n$ be an arithmetically Gorenstein line arrangement of Castelnuovo-Mumford regularity $r + 1$. Then:

(i) (Benedetti-_, 2014) $G(X)$ is r-connected.

(ii) (Benedetti-Di Marca-_, 2016) If furthermore X has only planar singularities, then $G(X)$ is also r-regular.

Sketch of the proof

Let $X_1, \ldots, X_s$ be the irreducible components (lines) of X.

- $S = \mathbb{k}[x_0, \ldots, x_n]$;
- for $j = 1, \ldots, s$, $I_j \subseteq S$ the saturated ideal defining $X_j \subseteq \mathbb{P}^n$;
- $I_X = \bigcap_{j=1}^{s} I_j \subseteq S$ the (saturated) ideal defining $X \subseteq \mathbb{P}^n$;
- for any subset $A \subseteq \{1, \ldots, s\}$, $X_A = \bigcup_{j \in A} X_j \subseteq \mathbb{P}^n$.

Sketch of the proof

(i): Given a subset $A \subseteq \{1, \ldots, s\}$, set $B = \{1, \ldots, s\} \setminus A$. Because X_A and X_B are linked by $X \subseteq \mathbb{P}^n$, which is aGorenstein:

$$H^1(X_A, \mathcal{I}_{X_A}(k)) \cong H^1(X_B, \mathcal{I}_{X_B}(r-2-k)) \;\; \forall\, k \in \mathbb{Z}.$$

Derksen-Sidman: $H^1(X_A, \mathcal{I}_{X_A}(k)) = 0$ for all $k \geq |A| - 1$.

So, whenever $|A| < r$, $H^1(X_B, \mathcal{I}_{X_B}) = 0$ i.e. $H^0(X_B, \mathcal{O}_{X_B}) \cong \Bbbk$. So X_B is connected whenever $|A| < r$; i.e. $G(X)$ is r-connected.

(ii): Let d be the valency of the vertex s of $G(X)$; set $J = \bigcap_{j=1}^{s-1} I_j$ and $K = I_s + J$. Since X has only planar singularities, one sees that

$$K = I_s + (f)$$

where $f \in S$ is a homogeneous polynomial of degree d. So $\mathrm{Tor}_n^S(S/K, \Bbbk)$ is not zero in degree $n + d - 1$.

By the short exact sequence $0 \to S/I_X \to S/I_s \oplus S/J \to S/K \to 0$, we get the long exact sequence of graded S-modules

$$\ldots \to \mathrm{Tor}^S_n(S/I_s, \Bbbk) \oplus \mathrm{Tor}^S_n(S/J, \Bbbk) \to \mathrm{Tor}^S_n(S/K, \Bbbk) \to \mathrm{Tor}^S_{n-1}(S/I_X, \Bbbk) \to \ldots$$

Since $X_{\{1,2,\ldots,s-1\}}$ is linked to X_s by X, we have

$$\mathrm{Tor}^S_n(S/J, \Bbbk) = \mathrm{Tor}^S_n(S/I_s, \Bbbk) = 0.$$

Therefore the map $\mathrm{Tor}^S_n(S/K, \Bbbk) \to \mathrm{Tor}^S_{n-1}(S/I_X, \Bbbk)$ is injective, so that $\mathrm{Tor}^S_{n-1}(S/I_X, \Bbbk)$ is not zero in degree $n - 1 + d$ as well. So

$$r + 1 = \operatorname{reg} X \geq d + 1.$$

On the other hand we proved in (i) that $G(X)$ is r-connected, so

$$d \geq r.$$

Corollary

Let $\iota_{|H|} : Z \hookrightarrow \mathbb{P}^n$ be a smooth surface, and let $L_1, \ldots, L_s$ be lines on it such that $L_1 + L_2 + \ldots + L_s \sim dH$.

(i) If $n = 3$, then each L_i intersects exactly $\deg X + d - 2$ among the other L_j's.

(ii) If Z is a K_3 surface, then each L_i intersects exactly $d + 2$ among the other L_j's.

Open questions

Let $\iota_{|H|} : Z \hookrightarrow \mathbb{P}^3_{\mathbb{C}}$ be a smooth surface of degree d. We already discussed the cases $d \leq 3$. When $d = 4$, if Z contains the maximum possible number of lines (which is 64 by Rams and Schütt), the sum of these lines is a divisor of Z linearly equivalent to $16H$. Is there something more general behind this?

Let $X \subseteq \mathbb{P}^3$ be a line arrangement. If X is the complete intersection of two surfaces of degree d and e then, as a consequence of Theorem (i), $G(X)$ has diameter at most $\min\{d, e\}$. On the other hand, the larger diameter I know for the dual graph of a c.i. line arrangement in $\mathbb{P}^3$ is 3, attained by Schläfli double six. Is the diameter of a c.i. line arrangement in $\mathbb{P}^3$ bounded?

THANKS FOR YOUR ATTENTION !!!