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On: 20 July 2011, At: 09:19

Publisher: Taylor & Francis

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Communications in Algebra

Publication details, including instructions for authors and subscription information:

<http://www.tandfonline.com/loi/lagb20>

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Bruno Benedetti ^a & Matteo Varbaro ^b

^a Institut für Mathematik, FU Berlin, Germany

^b Dipartimento di Matematica, Università degli Studi di Genova, Italy

Available online: 20 Jul 2011

To cite this article: Bruno Benedetti & Matteo Varbaro (2011): Unmixed Graphs that are Domains, Communications in Algebra, 39:7, 2260-2267

To link to this article: <http://dx.doi.org/10.1080/00927872.2010.519363>

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UNMIXED GRAPHS THAT ARE DOMAINS

Bruno Benedetti¹ and Matteo Varbaro²

¹Institut für Mathematik, FU Berlin, Germany

²Dipartimento di Matematica, Università degli Studi di Genova, Italy

We extend a theorem of Villarreal on bipartite graphs to the class of all graphs. On the way to this result, we study the basic covers algebra $\overline{A}(G)$ of an arbitrary graph G . We characterize with purely combinatorial methods the cases when 1) $\overline{A}(G)$ is a domain and 2) G is unmixed and $\overline{A}(G)$ is a domain.

Key Words: Basic covers; Minimal vertex covers; Symbolic powers; Unmixed graphs.

2000 Mathematics Subject Classification: 05C70; 13A30.

1. INTRODUCTION AND NOTATION

Fix a graph on n vertices and give each vertex a price; let the “cost” of an edge be the sum of the costs of its endpoints. A nonzero price distribution such that no edge is cheaper than k euros is called a k -cover.

A k -cover and a k' -cover of the same graph can be summed vertex-wise, yielding a $(k + k')$ -cover; one says that a k -cover is *basic* if it cannot be decomposed into the sum of a k -cover and a 0-cover. Basic 1-covers of a graph are also known as “minimal vertex covers” and have been extensively studied by graph theorists.

A graph G is called *unmixed* if all its basic 1-covers have the same number of ones. For example, a square is unmixed, a pentagon is unmixed, yet a hexagon is not unmixed. A graph G is called a *domain* if, for all $s, t \in \mathbb{N}$, any s basic 1-covers and any t basic 2-covers always add up to a basic $(s + 2t)$ -cover. For example, the square is a domain, while the pentagon and the hexagon are not domains.

This notation is motivated by the following algebraic interpretation (see Herzog et al. [2], or Benedetti et al. [1] for details). Let S be a polynomial ring of n variables over some field, and let \mathfrak{m} be its irrelevant ideal. Let $I(G)$ be the ideal of S generated by all the monomials $x_i x_j$ such that $\{i, j\}$ is an edge of G . The ideal $I(G)$ is called *edge ideal* of G , and its Alexander dual $J(G) = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j)$ is called *cover ideal* of G . The *symbolic fiber cone* of $J(G)$ is $\overline{A}(G) = R_s(J(G))/\mathfrak{m}R_s(J(G))$, where $R_s(J(G)) = \bigoplus_{i \geq 0} J(G)^{(i)}$ is the symbolic Rees algebra of $J(G)$ and $J(G)^{(i)}$ is the i th symbolic power of $J(G)$.

Received September 23, 2009. Communicated by R. Villarreal.

Address correspondence to Dr. Matteo Varbaro, Dipartimento di Matematica, Università degli Studi di Genova, Via Dodecaneso 35, Italy; E-mail: varbaro@dima.unige.it; Dr. Bruno Benedetti, Inst. Mathematics, Arnimallee 2, FU Berlin, Germany; E-mail: benedetti@math.fu-berlin.de

The following facts are easy to prove:

- a) G is unmixed if and only if $I(G)$ is unmixed;
- b) G is a domain if and only if $\overline{A}(G)$ is a domain;
- c) if G is a domain, then $\overline{A}(G)$ is a Cohen–Macaulay algebra (and a normal domain).

In the present paper, we introduce three entirely combinatorial properties, called “square condition” (SC), “weak square condition” (WSC), and “matching square condition” (MSC).

Definition 1.1. We say that a graph G satisfies:

- i) **SC**, if for each triple of consecutive edges $\{i, i'\}$, $\{i, j\}$, and $\{j, j'\}$ of G , one has that $i' \neq j'$ and $\{i', j'\}$ is also an edge of G ;
- ii) **WSC**, if G has at least one edge, and for every non-isolated vertex i there exists an edge $\{i, j\}$ such that for all edges $\{i, i'\}$ and $\{j, j'\}$ of G , $\{i', j'\}$ is also an edge of G (in particular, $i' \neq j'$);
- iii) **MSC**, if the graph G^{red} obtained by deleting all the isolated vertices of G is nonempty, and admits a perfect matching such that for each edge $\{i, j\}$ of the matching, and for all edges $\{i, i'\}$ and $\{j, j'\}$ of G , one has that $\{i', j'\}$ is also an edge of G (in particular, $i' \neq j'$).

We will see in Lemma 2.3 that the first property is satisfied only by bipartite complete graphs, by isolated vertices, and by disjoint unions of these two types of graphs.

The second property, WSC, is a weakened version of the first one. It was studied in [1], where the authors proved that when G is bipartite, G satisfies WSC if and only if G is a domain. We extend this result to non-bipartite graphs.

Main Theorem 1 (Theorem 2.5). *A graph G satisfies WSC if and only if G is a domain.*

Finally, the third property was investigated by Villarreal [3, Theorem 1.1], who proved that, when G is bipartite, G satisfies MSC if and only if G is unmixed. In the present paper, we extend Villarreal’s theorem to the non-bipartite case, showing the following theorem.

Main Theorem 2 (Theorem 2.8). *A graph G satisfies MSC if and only if G is an unmixed domain.*

This implies Villarreal’s result because in the bipartite case all unmixed graphs are domains. However, many graphs satisfy MSC without being bipartite (see Theorem 2.10). From an algebraic point of view, Main Theorem 2 characterizes the graphs G for which every symbolic power of the cover ideal of G is generated by monomials of the same degree.

We point out that the proof of Main Theorem 2 is not an extension of Villarreal’s proof. We follow a different approach, introducing the graph G^{0-1} , which is obtained from G by removing the isolated vertices and then by removing all

edges $\{i, j\}$ such that there exists a basic 1-cover \mathbf{a} of G for which $a_i + a_j = 2$. This graph G^{0-1} always satisfies SC (see Lemma 2.4). Furthermore, G^{0-1} has no isolated points if and only if the original graph G satisfied WSC (see Theorem 2.5); finally, G^{0-1} admits a matching if and only if G satisfies MSC (see Theorem 2.8).

For example, let G be the graph on six vertices, given by the edges $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 4\}$, $\{2, 5\}$, $\{4, 5\}$, and $\{5, 6\}$. This graph G has three basic 1-covers. As G^{0-1} is the disjoint union of a $K_{2,2}$ and a $K_{1,1}$, we have that G is an unmixed domain and satisfies MSC.

2. PROOFS OF THE MAIN THEOREMS

Lemma 2.1. *Let $\{i, j\}$ be an edge of a graph G . For any integer $d \geq 1$, the following are equivalent:*

- (1) *For any $k \in \{1, \dots, d\}$, and for any basic k -cover \mathbf{a} , $a_i + a_j = k$;*
- (2) *For any basic 1-cover \mathbf{a} , $a_i + a_j = 1$;*
- (3) *If $\{i, i'\}$ is an edge of G and $\{j, j'\}$ an edge of G , then $i' \neq j'$ and $\{i', j'\}$ is also an edge of G .*

Proof. (2) is a special case of (1). To see that (2) implies (3), we argue by contradiction. If G contains a triangle $\{i, i'\}$, $\{i, j\}$, $\{j, i'\}$, we claim that there is a basic 1-cover \mathbf{a} such that $a_i = a_j = 1$ and $a_{i'} = 0$. In fact, define a 1-cover \mathbf{b} by setting $b_{i'} = 0$, and $b_k = 1$ for all $k \neq i'$. In case \mathbf{b} is basic we are done; otherwise, \mathbf{b} breaks into the sum of a basic 1-cover \mathbf{a} and some 0-cover. This \mathbf{a} has still the property of yielding 0 on i' , and thus 1 on i and j , so we are done. If instead G contains four edges $\{i, i'\}$, $\{i, j\}$ and $\{j, j'\}$, but not the fourth edge $\{i', j'\}$, we claim that there is a basic 1-cover \mathbf{a} such that $a_i = a_j = 1$ and $a_{i'} = a_{j'} = 0$. The proof is as before: First one defines a 1-cover \mathbf{b} by setting $b_{i'} = b_{j'} = 0$, and $b_k = 1$ for all k such that $j' \neq k \neq i'$; then one reduces \mathbf{b} to a basic 1-cover.

Finally, assume (1) is false: Then there is a basic k -cover \mathbf{a} such that $a_i + a_j > k$. For the cover to be basic, there must be a neighbour i' of i such that $a_{i'} + a_i = k$, and a neighbour j' of j such that $a_{j'} + a_j = k$. But then $a_{i'} + a_{j'} = 2k - a_i - a_j < k$, so $\{i', j'\}$ cannot be an edge of G : Hence, (3) is false, too. Thus (3) implies (1). \square

In the proof of the next Lemma we use a convenient shortening: we say that a k -cover \mathbf{a} can be “lopped at the vertex i ” if replacing a_i with $a_i - 1$ in the vector \mathbf{a} still yields a k -cover.

Lemma 2.2. *Let G be a graph. G is a domain if and only if G has at least one edge, and for each non-isolated vertex i there exists a vertex j adjacent to i in G such that:*

- a) *For any basic 1-cover \mathbf{a} one has $a_i + a_j = 1$, and*
- b) *For any basic 2-cover \mathbf{b} one has $b_i + b_j = 2$.*

Proof. The fact that G is a domain rules out the possibility that G might be a disjoint union of points; so let us assume that G has at least one edge. G is not a domain if and only if a nonbasic $(s + 2t)$ -cover of G can be written as the sum

of s basic 1-covers and t basic 2-covers, if and only if there is a vertex i such that a certain sum \mathbf{c} of s basic 1-covers and t basic 2-covers can be lopped at the vertex i (and, in particular, this i cannot be isolated), if and only if there exists a non-isolated vertex i such that, for each edge $\{i, j\}$, either there exists a basic 1-cover a such that $a_i + a_j > 1$, or there exists a basic 2-cover b such that $b_i + b_j > 2$. \square

Lemma 2.3. *Let G be a connected graph. G satisfies SC if and only if G is either a single point or a $K_{a,b}$, for some $b \geq a \geq 1$.*

Proof. The fact that a $K_{a,b}$ satisfies SC is obvious. For the converse implication, first note that a graph G satisfying SC cannot contain triangles; moreover, if G contained a $(2d + 1)$ -cycle, by SC we could replace three edges of this cycle by a single edge, hence G would contain a $(2d - 1)$ -cycle as well. By induction on d we conclude that G contains no odd cycle. So G is bipartite: If $[n] = A \cup B$ is the bipartition of its vertices, we claim that any vertex in A is adjacent to any vertex in B . In fact, if G has no three consecutive edges, then G is either a point or a $K_{1,b}$ (for some positive integer b) and there is nothing to prove. Otherwise, take $a \in A$ and $b \in B$: Since G is connected, there is an (odd length) path from a to b . By SC, the first three edges of such a path can be replaced by a single edge, yielding a path that is two steps shorter. Iterating the trick, we eventually find a path of length 1 (that is, an edge) from a to b . \square

Definition 2.1. Let G be a graph with at least one edge. We denote by G^{0-1} the graph that has:

- a) As vertices, the vertices of G^{red} ;
- b) As edges, the edges $\{i, j\}$ of G such that for every basic 1-cover \mathbf{a} of G one has $a_i + a_j = 1$.

Lemma 2.4. *Let G be an arbitrary graph with at least one edge. Then G^{0-1} satisfies SC.*

Proof. Assume $\{h, i\}$, $\{i, j\}$ and $\{j, k\}$ are three consecutive edges of G^{0-1} . For any basic 1-cover \mathbf{a} , $a_h + a_i = 1$, $a_i + a_j = 1$ and $a_j + a_k = 1$. Summing up the three equations, we obtain that $a_h + 2a_i + 2a_j + a_k = 3$, thus $a_h + a_k = 1$: so all we need to prove is that $\{h, k\}$ is an edge of G . But by Lemma 2.1, this follows from the fact that for any basic 1-cover \mathbf{a} one has $a_i + a_j = 1$. \square

Theorem 2.5. *Let G be a graph with at least one edge. Then the following are equivalent:*

- (1) G satisfies WSC;
- (2) \overline{G} is a domain;
- (3) $\overline{A}(G)$ is a domain;
- (4) G^{0-1} has no isolated points.

Proof. The equivalence of (1) and (2) follows from combining Lemma 2.1 and Lemma 2.2. The equivalence of (2) and (3) was explained in the Introduction. The equivalence of (1) and (4) is straightforward from Lemma 2.1. \square

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Lemma 2.6. *Let G be a domain. Let H_1, \dots, H_k be the connected components of G^{0-1} ; let $A_i \cup B_i$ be the bipartition of the vertices of H_i , for $i = 1, \dots, k$. Then:*

- (1) *If G contains a triangle, all three edges are not in G^{0-1} ; in particular, two vertices of the same B_i (or of the same A_i) are not adjacent in G ;*
- (2) *If G contains an edge between a vertex of A_i and a vertex of A_j , then it contains also edges between any vertex of A_i and any vertex of A_j ;*
- (3) *If G contains an edge between A_i and A_j , then it contains no edge between B_i and B_j ;*
- (4) *If G contains an edge between A_i and A_j and another edge between B_i and B_k , then it contains an edge between A_j and B_k ;*
- (5) *If G contains an edge between A_h and A_i and another edge between A_h and A_j , then G contains no edge between B_i and B_j .*

Proof. (1) Choose a vertex v of the triangle, and take a basic 1-cover that yields 0 on v . This 1-cover yields 1 on the other two vertices, so the edge opposite to v does not belong to G^{0-1} . The second part of the claim follows from the fact that each H_i is complete bipartite: Were two adjacent vertices both in A_i (or both in B_i), they would have a common neighbour in G^{0-1} , so there would be a triangle in G with two edges in G^{0-1} , a contradiction.

(2) Take a vertex i of A_i adjacent in G to a point j of A_j . Let i' be any point of A_i different from i . By contradiction, there is a vertex j' of A_j that is not adjacent to i' . Construct a basic 1-cover \mathbf{c} that yields 0 on i' , and 0 on A_j (\mathbf{c} is well defined because no two vertices of A_j can be adjacent, by the previous item). Since $c_j = 0$, c_i must be 1; and since $c_{i'} = 0$, \mathbf{c} yields 1 on B_i ; but then all edges $\{i, b\}$ with $b \in B_i$ are not in G^{0-1} , a contradiction.

(3) Assume G contains an edge $\{i, j\}$ between A_i and A_j , and choose any vertex x of B_i , and any vertex y of B_j . Take a basic 1-cover \mathbf{a} such that $a_i + a_j = 2$. Since $\{x, i\}$ and $\{y, j\}$ are in G^{0-1} , $a_i + a_x = a_j + a_y = 1$; thus $a_x = a_y = 0$, which implies that there cannot be an edge in G from x to y .

(4) Fix a vertex i of A_i and use the WSC property (which G satisfies by Theorem 2.5): There exists a x adjacent to i such that for any edge $\{i, j\}$ and for any edge $\{x, y\}$ of G , $\{j, y\}$ is also an edge of G . By Lemma 2.1, $a_i + a_x = 1$ for each basic 1-cover \mathbf{a} ; that is to say, $\{i, x\}$ is in G^{0-1} . This implies that x is in B_i . So if G contains an edge $\{i, j\}$, with $j \in A_j$, and an edge $\{x, y\}$, with y in some B_k , then G contains also the edge $\{j, y\}$ from A_j to B_k .

(5) By contradiction, assume there is an edge between B_i and B_j . By the previous item, since there is an edge between A_h and A_j , there is also an edge between A_h and B_i ; but this contradicts the item (1), since there is an edge between A_h and A_i . \square

Lemma 2.7. *Let G be a domain. Let H be a single connected component of G^{0-1} , and let $A \cup B$ be the bipartition of the vertices of H . There exists a basic 1-cover \mathbf{a} of G that*

yields 1 on A , 0 on B , and such that the cover \mathbf{b} defined as

$$b_i = \begin{cases} 1 - a_i & \text{if } i \in H, \\ a_i & \text{otherwise} \end{cases}$$

is a basic 1-cover that yields 1 on B and 0 on A .

Proof. Let H_1, \dots, H_k denote the other connected components of G^{0-1} , and let $A_i \cup B_i$ be the bipartition of the vertices of H_i . By Lemma 2.6 [item (1)], no two points in B_i are adjacent. If in G there is an edge from A to some A_i , then B_i is not connected to B by any edge (cf. Lemma 2.6, item (3)); if in addition there are edges from A to some A_j with $j \neq i$, by 2.6, item (5), there is no edge between B_i and B_j either. Therefore, the vector that yields:

- a) 0 on B ,
- b) 0 on all the B_i 's such that A_i is connected with an edge to A , and
- c) 1 everywhere else,

is a 1-cover of G . If this 1-cover is basic, call it \mathbf{a} ; otherwise, decompose it into the sum of a basic 1-cover \mathbf{a} and a 0-cover \mathbf{c} . In any case, \mathbf{a} satisfies the desired properties. □

Definition 2.2. By *norm* of a k -cover we mean the sum of all its entries. We denote this as $|a| := \sum_{i=1}^n a_i$.

Theorem 2.8. Let G be a graph with n vertices, all of them non-isolated. Then the following are equivalent:

- (1) G satisfies MSC;
- (2) Every basic k -cover of G has norm $\frac{kn}{2}$;
- (3) For any k , the norm of all basic k -covers of G is the same;
- (4) G is an unmixed domain;
- (5) $\bar{A}(G)$ is a domain, and $I(G)$ is unmixed;
- (6) Every connected component of G^{0-1} is a $K_{a,a}$, for some positive integer a ;
- (7) G^{0-1} admits a matching;
- (8) G admits a matching, and all the basic 1-covers of G have exactly $\frac{n}{2}$ ones.

Proof. (1) \Rightarrow (2) The matching consists of $\frac{n}{2}$ edges, so if we show that for every edge $\{i, j\}$ of the matching and for every basic k -cover one has $a_i + a_j = k$, we are done. But this follows from Lemma 2.1, since $\{i', j'\}$ must be an edge of G whenever $\{i, i'\}$ and $\{j, j'\}$ are edges of G .

(2) \Rightarrow (3) Obvious.

(3) \Rightarrow (4) Setting $k = 1$ we get the definition of unmixedness. Now, denote by $f(k)$ the norm of any basic k -cover. Since twice a basic 1-cover yields a basic 2-cover, $2 \cdot f(1) = f(2)$; and in general $k \cdot f(1) = f(k)$. Suppose that an $(s + 2t)$ -cover \mathbf{a} can be written as the sum of s basic 1-covers and t basic 2-covers. The norm of \mathbf{a} can be computed via its summands, yielding $|\mathbf{a}| = s \cdot f(1) + t \cdot f(2) = (s + 2t) \cdot f(1)$.

Were \mathbf{a} non-basic, it could be written as the sum of a basic $(s + 2t)$ -cover \mathbf{b} and a 0-cover \mathbf{c} , whence $|\mathbf{a}| = |\mathbf{b}| + |\mathbf{c}| \geq |\mathbf{b}| + 1 = f(s + 2t) + 1 = (s + 2t) \cdot f(1) + 1$, a contradiction. This proves that G is a domain.

(4) \Leftrightarrow (5) Explained in the Introduction.

(4) \Rightarrow (6) Let H be a connected component of G^{0-1} . By Theorem 2.5, since G is a domain, H is bipartite complete; let $A \cup B$ be the bipartition of the vertices of H . Construct the basic 1-covers \mathbf{a} and \mathbf{b} as in Lemma 2.7; they have a different number of ones unless $|A| = |B|$, so by unmixedness we conclude.

(6) \Rightarrow (7), (6) \Rightarrow (8) Obvious.

(7) \Rightarrow (6) By Lemmas 2.3 and 2.4 every connected component of G^{0-1} is either a point or a bipartite complete graph; in order for G^{0-1} to admit a matching, each connected component of G^{0-1} must be of the form $K_{a,a}$, for some integer a .

(8) \Rightarrow (1) Let \mathbb{M} be the given matching. In view of Lemma 2.1, we only need to show that for every edge $\{i, j\}$ of \mathbb{M} and for every basic 1-cover \mathbf{a} one has $a_i + a_j = 1$. Yet for any basic 1-cover \mathbf{a} one has

$$\frac{n}{2} = \sum_{i=1}^n a_i = \sum_{\{i,j\} \in \mathbb{M}} a_i + a_j,$$

and a sum of $\frac{n}{2}$ positive integers equals $\frac{n}{2}$ only if each summand equals 1. \square

Corollary 2.9 (Villarreal). *Let G be a bipartite graph without isolated points. G is unmixed if and only if G satisfies MSC.*

Proof. In the bipartite case, “unmixed” implies “domain” (see, e.g., [1, 2]). So the condition (3) in Theorem 2.8 is equivalent to unmixedness. \square

The next result shows how to produce many examples of graphs (not necessarily bipartite) that satisfy the assumptions above.

Theorem 2.10. *Let G be an arbitrary graph.*

- a) *Let G^+ be the graph obtained by attaching a pendant to each vertex of G . Then G^+ satisfies MSC. Moreover, G^+ is bipartite if and only if G is bipartite.*
- b) *Let G' be the graph obtained from G attaching a pendant to each of those vertex of G that appear as isolated vertices in G^{0-1} . Then G' satisfies WSC. Moreover, G' satisfies MSC if and only if G^{0-1} is unmixed.*

Proof. Let us show the second item first. By definition of G^{0-1} (and by Lemma 2.1), the isolated vertices of G^{0-1} are exactly the vertices of G at which the WSC property does not hold. By attaching a pendant j at the vertex i , the property “if $\{i, i'\}$ and $\{j, j'\}$ are edges, then $\{i', j'\}$ is also an edge” holds true trivially, since j' must coincide with i .

Of course, in a matching of G' each pendant should be paired with the vertex it was attached to. By Theorem 2.8, (G') satisfies MSC if and only if $(G')^{0-1}$ has a matching, if and only if the graph obtained removing all isolated vertices from

G^{0-1} has a matching. This happens if and only if each connected component of G^{0-1} is either a single point or a $K_{a,a}$, for some positive integer a . This characterizes unmixedness within the class of graphs satisfying the SC property.

To prove the first item, label $1^+, 2^+, \dots, n^+$ the pendants attached to $1, 2, \dots, n$, respectively. The requested matching is $\{1, 1^+\}, \{2, 2^+\}, \dots, \{n, n^+\}$. The possible presence of an odd-cycle in G reflects in the presence of the same odd cycle in G' . \square

Examples and Remarks

1. The complete graph $G = K_4$ is unmixed and has a matching, but it does not satisfy MSC (it is not even a domain in fact). Every 1-cover has three ones, while $\frac{n}{2} = 2$. Note that the property “all basic 1-covers have norm $\frac{n}{2}$ ” is strictly stronger than unmixedness, while the property “for each k , all basic k -covers have norm $\frac{kn}{2}$ ” is equivalent to “for each k , the norm of all basic k -covers is the same.”

2. In Theorem 2.8, the assumption that no vertex is isolated was introduced only to simplify the notation. In general, an arbitrary graph G (with at least one edge) is an unmixed domain if and only if the reduced graph G^{red} , obtained by deleting from G the isolated points, is an unmixed domain. Clearly, the basic 1-covers of G will have exactly $\frac{|G^{red}|}{2}$ ones, and so on.

3. In view of Proposition 2.10, one might think that attaching pendants will make it more likely for a given graph to be a domain. However, let G be the graph with edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}$, and $\{2, 5\}$ (a square with a pendant attached). This G is a domain, yet if we attach a pendant to the vertex 3 the resulting graph is not a domain.

4. A basic 2-cover that cannot be the sum of two 1-covers is said to be *indecomposable*. Bipartite graphs have no indecomposable 2-covers [2], so in some sense the number of indecomposable 2-covers of a graph measures its “distance” from being bipartite. Suppose G contains an odd cycle and a vertex i none of whose neighbours is part of the cycle. One can see then that G admits a basic 2-cover \mathbf{a} that yields 0 on i and 1 on the cycle; such an \mathbf{a} is indecomposable, i.e., it cannot be the sum of two 1-covers. Now, the property of containing an odd cycle and a “distant” vertex is clearly inherited by G^+ , which satisfies MSC. This way one can see that the distance of an unmixed domain from being bipartite can be arbitrarily large.

ACKNOWLEDGMENTS

We would like to thank Aldo Conca and Jürgen Herzog for the kind support and encouragement.

The first author was supported by DFG via the Berlin Mathematical School.

REFERENCES

- [1] Benedetti, B., Constantinescu, A., Varbaro, M. (2008). Dimension, depth and zero-divisors of the algebra of basic k -covers of a graph. *Le Matematiche* LXIII: 117–156.
- [2] Herzog, J., Hibi, T., Trung, N. V. (2007). Symbolic powers of monomial ideals and vertex cover algebras. *Adv. Math.* 210:304–322.
- [3] Villarreal, R. H. (2007). Unmixed bipartite graphs. *Revista Colombiana de Matemáticas* 41:393–395.