CONNECTIVITY OF PSEUDOMANIFOLD GRAPHS FROM AN ALGEBRAIC POINT OF VIEW

KARIM A. ADIPRASITO, AFSHIN GOODARZI, AND MATTEO VARBARO

Abstract. The connectivity of graphs of simplicial and polytopal complexes is a classical subject going back at least to Steinitz, and the topic has since been studied by many authors, including Balinski, Barnette, Athanasiadis and Björner. In this note, we provide a unifying approach which allows us to obtain more general results. Moreover, we provide a relation to commutative algebra.

1. Connectivity of the underlying graph

Let Δ be a finite simplicial complex on the vertex set $[n] := \{1, \ldots, n\}$. The **underlying graph** (or **1-skeleton**) G_Δ of Δ is the graph obtained by restricting Δ to faces of cardinality at most two.

A graph G is said to be **k-connected** if it has at least k vertices and removing any subsets of vertices of cardinality less than k results in a connected graph. The **(vertex-)connectivity** κ_G of G is the maximum number k such that G is k-connected.

The classical Steinitz's theorem [Ste22] asserts that a graph G is the underlying graph of a 3-polytope if and only if G is 3-connected and planar. In 1961, Balinski extended the "only if" direction of Steinitz's theorem by showing that the underlying graph of a d-polytope is d-connected, cf. [Zie95]. David Barnette showed that the same bound is also valid for the connectivity number of underlying graphs of $(d-1)$-dimensional pseudomanifolds [Bar82].

Athanasiadis [Ath11] showed that if the pseudomanifold is also flag (i.e. the clique complex of its 1-skeleton), then this lower bound can be improved to $2d-2$. Björner and Vorwerk quantified this connection using the notion of banner complexes [BV14].

The purpose of this note is to provide a unifying approach which allows us to obtain more general results.

The proof is inspired by a relation of connectivity to the Hochster's formula (observed in [Goo14]) from commutative algebra and simple estimates for the size of certain flag complexes [ANT14].

2. Basics in commutative algebra

We start by recalling some notions, and refer to [MS05, HH11] for exact definitions and more details. Let I be a graded ideal in the polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ in n variables over a field $\Bbbk$. Let

$$\mathbf{F}_{S/I} \ := \ 0 \to F_p \to F_{p-1} \to \cdots \to F_1 \to F_0 \to S/I \to 0,$$

be the minimal graded free resolution of S/I, with $F_i = \bigoplus_j S(-j)^{b_{i,j}}$ in homological degree i. The number $b_{i,j} = b_{i,j}(S/I)$ is the **graded Betti number** of S/I in **homological degree** i and **internal degree** j. The length of the j-th row in the Betti table will be denoted by $lp_j(S/I)$, that is

$$lp_j(S/I) \ := \ \max\{i \mid b_{i,i+j-1}(S/I) \neq 0\}.$$

We also denote by $t_i(S/I)$ the **maximum internal degree** of a minimal generator in the homological degree i that is $\max\{j \mid b_{i,j} \neq 0\}$. The **projective dimension** of S/I is the maximum i such that $b_{i,j} \neq 0$, for some j. The **regularity** of S/I is defined to be $\max_i\{t_i(S/I) - i\}$.

3. Connectivity via graded Betti numbers

Let Δ be a simplicial complex on the vertex set $[n]$. The **Stanley–Reisner ideal** $I_\Delta \subset S$ of Δ is the ideal generated by monomials $\mathbf{x}_F := \prod_{i \in F} x_i$ for all F not in Δ. The quotient ring $\Bbbk[\Delta] = S/I_\Delta$ is called

---

Date: November 1, 2014.

K. Adiprasito was supported by an EPDI/IPDE postdoctoral fellowship and a Minerva fellowship of the Max Planck Society.

M. Varbaro was supported by PRIN 2010S47ARA_003 "Geometria delle Varietà Algebriche".

the **face ring** of Δ. In this case, **Hochster's formula** provides an interpretation of the graded Betti numbers in terms of the reduced homology of induced sub-complexes of Δ. More precisely, it asserts that

$$b_{i,j}(\Bbbk[\Delta]) \;=\; \sum_{\#W=j} \dim_{\Bbbk} \widetilde{H}_{j-i-1}(\Delta_W).$$

Recently it was observed by the second author [Goo14, Theorem 3.1] that using Hochster's formula, one can study the connectivity of a graph via homological invariants. The following result is an extension of this observation to underlying graphs of simplicial complexes.

Proposition 1. *Let Δ be a simplicial complex on the vertex set $[n]$ and κ be the connectivity number of its underlying graph. Then one has*

$$\kappa + lp_2(\Bbbk[\Delta]) = n - 1.$$

Proof. Let us denote by G the underlying graph of Δ. We also denote by $\mathrm{Cl}(G)$ the clique complex of G, i.e. the simplicial complex whose faces are the complete induced subgraphs (cliques) of G. Since $lp_2(-)$ depends only on the quadratic generators, one easily sees that

$$lp_2(\Bbbk[\Delta]) = lp_2(\Bbbk[\mathrm{Cl}(G)]).$$

On the other hand, Δ and $\mathrm{Cl}(G)$ both have the same underlying graph G. Hence, it suffices to verify the result in the case of flag complexes and we may assume that $\Delta = \mathrm{Cl}(G)$. Now the result follows from [Goo14, Theorem 3.1]. □

Before presenting our next result, we shall introduce two properties.

Definition 2. Let I be a graded ideal such that S/I is of regularity r. Set $m = lp_r(S/I)$. We say S/I satisfies the **property** $\mathfrak{A}$ if
(1) $lp_2(S/I) \le m$,
(2) $b_{m-i,m-i+1}(S/I) \le b_{i,i+r-1}(S/I)$.
We also say that S/I satisfies the **property** $\mathfrak{B}_s$ if for all $i < s$ one has $t_i(S/I) < r + i - 1$.

Remark 3. Poincaré duality and the Koszul property
(1) If S/I is Gorenstein, then it satisfies the property $\mathfrak{A}$. However, the property only requires a much simpler property than Poincaré–Lefschetz duality; a simple inequality shall be enough, see Lemma 8.
(2) If I is generated by quadratic monomials, then it is easy to see that it $t_s(S/I) \le 2s$ for all s and therefore S/I satisfies $\mathfrak{B}_{r-1}$. This fact is valid more generally when S/I is Koszul as was shown by Backelin in [Bac88], see also Kempf [Kem90].

Proposition 4. *Let I be a graded ideal in polynomial ring S. Moreover, assume that S/I has regularity r and satisfies the properties $\mathfrak{A}$ and $\mathfrak{B}_s$. Then one has*

$$s \le lp_r(S/I) - lp_2(S/I).$$

Proof. We have

$$\begin{aligned} lp_r(S/I) - lp_2(S/I) &= m - \max\{j \mid b_{j,j+1}(S/I) \neq 0\} \\ &= \min\{m - j \mid b_{j,j+1}(S/I) \neq 0\} \\ &\geq \min\{m - j \mid b_{m-j,m-j+r-1}(S/I) \neq 0\} && \text{(Property } \mathfrak{A}) \\ &= \min\{k \mid b_{k,k+r-1}(S/I) \neq 0\} \\ &\geq \min\{k \mid t_k(S/I) \geq k + r - 1\} \end{aligned}$$

where the last term is at least s by Property $\mathfrak{B}_s$. □

By Bakelin's result the regularity of a Koszul ring is bounded above by its projective dimension. As an immediate consequence of the previous result, we get the following tight bound for Gorenstein Koszul rings.

Corollary 5. *The regularity of a Gorenstein Koszul ring S/I is at most $\mathrm{projdim}(S/I) - lp_2(S/I) + 1$.* □

Theorem 6. *Let Δ be a $(d-1)$-dimensional simplicial complex with nontrivial top-homology. Also, assume that $\Bbbk[\Delta]$ satisfies the properties $\mathfrak{A}$ and $\mathfrak{B}_s$. Then the underlying graph is $(d+s-1)$-connected.*

Proof. Note that the regularity of $\Bbbk[\Delta]$ is equal to d since Δ has nontrivial top-homology. So, it follows from Proposition 4 that

$$s \le lp_d(\Bbbk[\Delta]) - lp_2(\Bbbk[\Delta]).$$

Note that $lp_d(\Bbbk[\Delta]) = n - d$. So, by Corollary 1 we get

$$s \leq n - d - (n - \kappa_\Delta - 1),$$

where κ_Δ stands for the connectivity number of the underlying graph of Δ. Therefore

$$\kappa_\Delta \geq d + s - 1. \qquad \square$$

Remark 7. As a special case of Theorem 6, we can consider Δ to be Gorenstein*. Then $\Bbbk[\Delta]$ satisfies the property $\mathfrak{B}_1$. Moreover, if Δ is also flag, then it satisfies the property $\mathfrak{B}_{d-1}$, since $t_i(\Bbbk[\Delta]) \leq 2i$ for any i and $\Bbbk[\Delta]$ is d-regular.

4. A Poincaré–Lefschetz-type inequality for minimal cycles

Recall that a **minimal d-cycle** Σ (w.r.t. a coefficient ring R) is a pure d-dimensional complex that supports precisely one homology d-class ζ whose support is the complex itself. For instance, every pseudomanifold is a minimal cycle (over $\mathbb{Z}/2\mathbb{Z}$); and so is every triangulation of a closed, connected manifold.

Lemma 8. *Let Σ denote any minimal d-cycle and W a subset of the vertex-set* $\mathrm{V}(\Sigma)$. *Then*

$$\mathrm{rk}\ \widetilde{H}_0(\Sigma_W) \ \leq\ \mathrm{rk}\ \widetilde{H}_{d-1}(\Sigma_{V(\Sigma)\setminus W})$$

Proof. Since Σ supports a global d-cycle (by minimality), we have an injection

$$H^0(\Sigma_W) \ \hookrightarrow\ H_d(\Sigma, \Sigma \setminus \Sigma_W).$$

To see this, notice that the restriction of the global d-cycle to any connected component of Σ_W induces a relative cycle for $(\Sigma, \Sigma \setminus \Sigma_W)$.

Now, since $\Sigma_{V(\Sigma)\setminus W}$ is homotopically equivalent to $\Sigma \setminus \Sigma_W$, the exact sequence

$$0 \ \longrightarrow\ \widetilde{H}_d(\Sigma) \ \longrightarrow\ \widetilde{H}_d(\Sigma, \Sigma_{V(\Sigma)\setminus W}) \ \longrightarrow\ \widetilde{H}_{d-1}(\Sigma_{V(\Sigma)\setminus W}) \ \longrightarrow\ \cdots,$$

implies

$$\begin{aligned} &\mathrm{rk}\ \widetilde{H}_{d-1}(\Sigma_{V(\Sigma)\setminus W}) + 1 \\ =\ &\mathrm{rk}\ \widetilde{H}_{d-1}(\Sigma_{V(\Sigma)\setminus W}) + \mathrm{rk}\ \widetilde{H}_d(\Sigma) \\ \geq\ &\mathrm{rk}\ \widetilde{H}_d(\Sigma, \Sigma_{V(\Sigma)\setminus W}) \\ \geq\ &\mathrm{rk}\ H^0(\Sigma_W). \end{aligned} \qquad \square$$

5. Applications to connectivity of manifolds

Let Δ be a $(d-1)$-dimensional simplicial complex on the vertex set $\mathrm{V}(\Delta)$. Recall the notion of banner complexes of [BV14]:

- A subset W of $\mathrm{V}(\Delta)$ is called **complete** if every two vertices of W form an edge of Δ.
- A complete set $W \subseteq \mathrm{V}(\Delta)$ is **critical** if $W \setminus \{v\}$ is a face of Δ for some $v \in W$.
- We say that Δ is **banner** if every critical complete set W of size at least d is a face of Δ.
- We define the **banner number** of Δ to be

$$\mathrm{b}(\Delta) \quad = \quad \min \left\{ b \quad : \quad \begin{array}{l} \mathrm{lk}_\sigma \Delta \text{ is banner or the boundary of the 2-simplex} \\ \text{for all faces } \sigma \in \Delta \text{ of cardinality } b \text{ and degree } d \end{array} \right\},$$

where the **degree** of a face is the maximal cardinality of a facet containing it.

Note that our notions of banner complexes and banner numbers are slightly more general then the ones introduced in [BV14]. However, if the complex is pure the definitions coincide.

Lemma 9. *Let Δ be a $(d-1)$-dimensional simplicial complex.*
(a) *If σ is a face of degree d in Δ, then* $\mathrm{b}(\mathrm{lk}_\sigma \Delta) \leq \max\{0, \mathrm{b}(\Delta) - \#\sigma\}$.
(b) *If Δ has nontrivial top-homology and* $\mathrm{b}(\Delta) < d - 2$, *then every induced subcomplex of Δ having nontrivial $(d-2)$-homology has at least* $2d - 2 - \mathrm{b}(\Delta)$ *vertices.*

Proof. The part (a) is clear from the definition. For claim (b), let us first show that, if Δ is banner, then every induced subcomplex Γ of Δ such that $\widetilde{H}_{d-2}(\Gamma) \neq 0$ has at least $2d-2$ vertices by induction on d. If $d = 3$, this is clear because Δ is flag.

Let $d > 3$. We may assume that no induced subcomplex of Δ has a nontrivial $(d-1)$-dimensional cycle: indeed, such a subcomplex is forced to have dimension $d-1$, so it would be banner and we could replace Δ with it. Furthermore, we may assume that Γ is a minimal induced subcomplex with the property that $\widetilde{H}_{d-2}(\Gamma) \neq 0$. Under such a minimality assumption, the link of any vertex of Γ admits a nontrivial homology cycle in dimension $d-3$. Take a vertex v of Γ. Since $\mathrm{lk}_v\Gamma$ is an induced subcomplex of $\mathrm{lk}_v\Delta$, which is banner and admits a nontrivial $(d-2)$-cycle, by induction $\mathrm{lk}_v\Gamma$ has at least $2d-4$ vertices. Moreover, v cannot be a cone point of Γ because $\widetilde{H}_{d-2}(\Gamma) \neq 0$, so $\#\mathrm{V}(\Gamma) \geq 2d-2$.

The claim (b) now follows by induction on the banner number and claim (a). □

Remark 10. While a flag simplicial complex (not necessarily of dimension $d-1$) supporting a nontrivial $(d-1)$-cycle has at least $2d$ vertices, this is false for banner complexes. Take the boundary of a d-simplex, and join one facet with an external edge: the resulting complex is a $(d+1)$-dimensional banner complex supporting a nontrivial $(d-1)$-cycle, but with only $d+3$ vertices.

Lemma 11. *Let Δ be a pure $(d-1)$-dimensional complex with nontrivial top-homology. If $\mathrm{b}(\Delta) < d-2$, then $\Bbbk[\Delta]$ satisfies the property $\mathfrak{B}_{d-\mathrm{b}(\Delta)-1}$.*

Proof. Notice that the regularity of $\Bbbk[\Delta]$ is d, since Δ has a nontrivial top-homology. If $b_{i,i+d-1}(\Bbbk[\Delta]) \neq 0$, by Hochster's formula there exists a subset $W \subseteq V(\Delta)$ of cardinality $i+d-1$ such that Δ_W supports a nontrivial $(d-2)$-cycle. By part (b) of Lemma 9, thus:

$$i \geq d - \mathrm{b}(\Delta) - 1.$$ □

Theorem 12. *Let Δ be an $(d-1)$-dimensional minimal cycle. Then the underlying graph of Δ is $(2d-\mathrm{b}(\Delta)-2)$-connected.*

Proof. If $\mathrm{b}(\Delta) = d-2$, then it is easy to see that $\Bbbk[\Delta]$ satisfies $\mathfrak{B}_1$. By Lemma 8 and Hochster's formula, $\Bbbk[\Delta]$ satisfies also the property $\mathfrak{A}$. Therefore, the result follows from Theorem 6.

If $\mathrm{b}(\Delta) < d-2$, by Lemmata 8 (together with Hochster's formula) and 11, $\Bbbk[\Delta]$ satisfies the properties $\mathfrak{A}$ and $\mathfrak{B}_{d-\mathrm{b}(\Delta)-1}$. Therefore, the result follows from Theorem 6. □

Corollary 13. *Let Δ be a flag (or more generally banner) $(d-1)$-dimensional minimal cycle. Then the underlying graph of Δ is $(2d-2)$-connected.*

Proof. If Δ is a banner complex, then $b(\Delta) = 0$. □

References

[ANT14] K. A. Adiprasito, E. Nevo, and M. Tancet, *Improved bounds for cohomological dimension and betti numbers of flag complexes*, in preparation. 1

[Ath11] C. A. Athanasiadis, *Some combinatorial properties of flag simplicial pseudomanifolds and spheres*, Ark. Mat. **49** (2011), no. 1, 17–29 (English). 1

[Bac88] J. Backelin, *Relations between rates of growth of homologies*, Research Reports in Mathematics, Mat. institutionen, Stockholms univ. **25** (1988). 2

[Bar82] D. Barnette, *Decompositions of homology manifolds and their graphs*, Israel J. Math. **41** (1982), no. 3, 203–212. 1

[BV14] A. Björner and K. Vorwerk, *On the connectivity of manifold graphs*, Proc. Am. Math. Soc. (2014) (English), to appear. 1, 3

[Goo14] A. Goodarzi, *Clique vectors of k-connected chordal graphs*, preprint, 8 pages, arXiv:1403.6210, 2014. 1, 2

[HH11] J. Herzog and T. Hibi, *Monomial ideals*, Graduate Texts in Mathematics, vol. 260, Springer-Verlag London Ltd., London, 2011. 1

[Kem90] G. R. Kempf, *Some wonderful rings on algebraic geometry*, J. Algebra **134** (1990), no. 1, 222–224. 2

[MS05] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. 1

[Ste22] E. Steinitz, *Polyeder und Raumeinteilungen*, Encyklopädie der mathematischen Wissenschaften, Dritter Band: Geometrie, III.1.2., Heft 9, Kapitel III A B 12 (W. Fr. Meyer and H. Mohrmann, eds.), B. G. Teubner, Leipzig, 1922, pp. 1–139. 1

[Zie95] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics, vol. 152, Springer, New York, 1995, Revised edition, 1998; seventh updated printing 2007. 1

Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France, *and* Einstein Institute for Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel

E-mail address: `adiprasito@ihes.fr, adiprssito@math.fu-berlin.de`

Department of Mathematics, Kungliga Tekniska Högskolan, S-100 44 Stockholm, Sweden

E-mail address: `afshingo@math.kth.se`

Dipartimento di Matematica, Università di Genova Via Dodecaneso 35-16146, Genova, Italy

E-mail address: `varbaro@dima.unige.it`