1. Connectivity of the underlying graph

Let $\Delta$ be a finite simplicial complex on the vertex set $[n] := \{1, \ldots, n\}$. The underlying graph (or 1-skeleton) $G_{\Delta}$ of $\Delta$ is the graph obtained by restricting $\Delta$ to faces of cardinality at most two.

A graph $G$ is said to be $k$-connected if it has at least $k$ vertices and removing any subsets of vertices of cardinality less than $k$ results in a connected graph. The (vertex-)connectivity $\kappa_G$ of $G$ is the maximum number $k$ such that $G$ is $k$-connected.

The classical Steinitz’s theorem [Ste22] asserts that a graph $G$ is the underlying graph of a 3-polytope if and only if $G$ is 3-connected and planar. In 1961, Balinski extended the “only if” direction of Steinitz’s theorem by showing that the underlying graph of a $d$-polytope is $d$-connected, cf. [Zie95]. David Barnette showed that the same bound is also valid for the connectivity number of underlying graphs of $(d - 1)$-dimensional pseudomanifolds [Bar82].

Athanasiadis [Ath11] showed that if the pseudomanifold is also flag (i.e. the clique complex of its 1-skeleton), then this lower bound can be improved to $2d - 2$. Björner and Vorwerk quantified this connection using the notion of banner complexes [BV14].

The purpose of this note is to provide a unifying approach which allows us to obtain more general results. The proof is inspired by a relation of connectivity to the Hochster’s formula (observed in [Goo14]) from commutative algebra and simple estimates for the size of certain flag complexes [ANT14].

2. Basics in commutative algebra

We start by recalling some notions, and refer to [MS05, HH11] for exact definitions and more details. Let $I$ be a graded ideal in the polynomial ring $S = \mathbb{k}[x_1, \ldots, x_n]$ in $n$ variables over a field $\mathbb{k}$. Let

$$F_{S/I} := 0 \to F_p \to F_{p-1} \to \cdots \to F_1 \to F_0 \to S/I \to 0,$$

be the minimal graded free resolution of $S/I$, with $F_i = \bigoplus_j S(-j)^{b_{i,j}}$ in homological degree $i$. The number $b_{i,j} = b_{i,j}(S/I)$ is the graded Betti number of $S/I$ in homological degree $i$ and internal degree $j$. The length of the $j$-th row in the Betti table will be denoted by $l_p(S/I)$, that is

$$l_p(S/I) := \max\{i \mid b_{i,i+j-1}(S/I) \neq 0\}.$$

We also denote by $t_i(S/I)$ the maximum internal degree of a minimal generator in the homological degree $i$ that is $\max\{j \mid b_{i,j} \neq 0\}$. The projective dimension of $S/I$ is the maximum $i$ such that $b_{i,j} \neq 0$, for some $j$. The regularity of $S/I$ is defined to be $\max\{t_i(S/I) - i\}$.

3. Connectivity via graded Betti numbers

Let $\Delta$ be a simplicial complex on the vertex set $[n]$. The Stanley–Reisner ideal $I_{\Delta} \subset S$ of $\Delta$ is the ideal generated by monomials $x_F := \prod_{i \in F} x_i$ for all $F$ not in $\Delta$. The quotient ring $\mathbb{k}[\Delta] = S/I_{\Delta}$ is called
the face ring of $\Delta$. In this case, Hochster’s formula provides an interpretation of the graded Betti numbers in terms of the reduced homology of induced sub-complexes of $\Delta$. More precisely, it asserts that

$$b_{i,j}(k[\Delta]) = \sum_{\#W = j} \dim_k \tilde{H}_{i-1}(\Delta_W).$$

Recently it was observed by the second author [Goo14, Theorem 3.1] that using Hochster’s formula, one can study the connectivity of a graph via homological invariants. The following result is an extension of this observation to underlying graphs of simplicial complexes.

**Proposition 1.** Let $\Delta$ be a simplicial complex on the vertex set $[n]$ and $\kappa$ be the connectivity number of its underlying graph. Then one has

$$\kappa + \lproj_2(k[\Delta]) = n - 1.$$

**Proof.** Let us denote by $G$ the underlying graph of $\Delta$. We also denote by $\text{Cl}(G)$ the clique complex of $G$, i.e. the simplicial complex whose faces are the complete induced subgraphs (cliques) of $G$. Since $\lproj_2(-)$ depends only on the quadratic generators, one easily sees that

$$\lproj_2(k[\Delta]) = \lproj_2(k[\text{Cl}(G)]).$$

On the other hand, $\Delta$ and $\text{Cl}(G)$ both have the same underlying graph $G$. Hence, it suffices to verify the result in the case of flag complexes and we may assume that $\Delta = \text{Cl}(G)$. Now the result follows from [Goo14, Theorem 3.1]. □

Before presenting our next result, we shall introduce two properties.

**Definition 2.** Let $I$ be a graded ideal such that $S/I$ is of regularity $r$. Set $m = \lproj_r(S/I)$. We say $S/I$ satisfies the property $\mathfrak{A}$ if

1. $\lproj_2(S/I) \leq m$,
2. $b_{n-i,m-1}(S/I) \leq b_{i+r+1}(S/I)$.

We also say that $S/I$ satisfies the property $\mathfrak{B}_s$ if for all $i < s$ one has $t_i(S/I) < r + i - 1$.

**Remark 3.** Poincaré duality and the Koszul property

1. If $S/I$ is Gorenstein, then it satisfies the property $\mathfrak{A}$. However, the property only requires a much simpler property than Poincaré–Lefschetz duality; a simple inequality shall be enough, see Lemma 8.
2. If $I$ is generated by quadratic monomials, then it is easy to see that it $t_s(S/I) \leq 2s$ for all $s$ and therefore $S/I$ satisfies $\mathfrak{B}_{r-1}$. This fact is valid more generally when $S/I$ is Koszul as was shown by Backelin in [Bac88], see also Kempf [Kem90].

**Proposition 4.** Let $I$ be a graded ideal in polynomial ring $S$. Moreover, assume that $S/I$ has regularity $r$ and satisfies the properties $\mathfrak{A}$ and $\mathfrak{B}_s$. Then one has

$$s \leq \lproj_r(S/I) - \lproj_2(S/I).$$

**Proof.** We have

$$\lproj_r(S/I) - \lproj_2(S/I) = m - \max\{j \mid b_{j,j+1}(S/I) \neq 0\} = \min\{m - j \mid b_{j,j+1}(S/I) \neq 0\} \geq \min\{m - j \mid b_{m-j,m-j+r-1}(S/I) \neq 0\} \geq \min\{s \mid t_s(S/I) \geq k + r - 1\}$$

(Property $\mathfrak{A}$)

where the last term is at least $s$ by Property $\mathfrak{B}_s$. □

By Bakelin’s result the regularity of a Koszul ring is bounded above by its projective dimension. As an immediate consequence of the previous result, we get the following tight bound for Gorenstein Koszul rings.

**Corollary 5.** The regularity of a Gorenstein Koszul ring $S/I$ is at most $\text{projdim}(S/I) - \lproj_2(S/I) + 1$. □

**Theorem 6.** Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex with nontrivial top-homology. Also, assume that $k[\Delta]$ satisfies the properties $\mathfrak{A}$ and $\mathfrak{B}_s$. Then the underlying graph is $(d + s - 1)$-connected.

**Proof.** Note that the regularity of $k[\Delta]$ is equal to $d$ since $\Delta$ has nontrivial top-homology. So, it follows from Proposition 4 that

$$s \leq \lproj_d(k[\Delta]) - \lproj_2(k[\Delta]).$$

Note that $\lproj_d(k[\Delta]) = n - d$. So, by Corollary 1 we get
\[ s \leq n - d - (n - \kappa_\Delta - 1), \]
where \( \kappa_\Delta \) stands for the connectivity number of the underlying graph of \( \Delta \). Therefore
\[ \kappa_\Delta \geq d + s - 1. \]

**Remark 7.** As a special case of Theorem 6, we can consider \( \Delta \) to be Gorenstein*. Then \( \kappa[\Delta] \) satisfies the property \( \mathfrak{B}_1 \). Moreover, if \( \Delta \) is also flag, then it satisfies the property \( \mathfrak{B}_{d-1} \), since \( t_i(\kappa[\Delta]) \leq 2t \) for any \( i \) and \( \kappa[\Delta] \) is \( d \)-regular.

4. A Poincaré–Lefschetz-type inequality for minimal cycles

Recall that a **minimal \( d \)-cycle** \( \Sigma \) (w.r.t. a coefficient ring \( R \)) is a pure \( d \)-dimensional complex that supports precisely one homology \( d \)-class \( \zeta \) whose support is the complex itself. For instance, every pseudomanifold is a minimal cycle (over \( \mathbb{Z}/2\mathbb{Z} \)); and so is every triangulation of a closed, connected manifold.

**Lemma 8.** Let \( \Sigma \) denote any minimal \( d \)-cycle and \( W \) a subset of the vertex-set \( V(\Sigma) \). Then
\[ \text{rk} \; \tilde{H}_0(\Sigma|_{\Sigma\setminus W}) \leq \text{rk} \; \tilde{H}_{d-1}(\Sigma|_{\Sigma\setminus W}) \]

**Proof.** Since \( \Sigma \) supports a global \( d \)-cycle (by minimality), we have an injection
\[ H^0(\Sigma|_{\Sigma\setminus W}) \hookrightarrow H_d(\Sigma, \Sigma \setminus \Sigma|_{\Sigma\setminus W}). \]

To see this, notice that the restriction of the global \( d \)-cycle to any connected component of \( \Sigma|_{\Sigma\setminus W} \) induces a relative cycle for \((\Sigma, \Sigma \setminus \Sigma|_{\Sigma\setminus W})\).

Now, since \( \Sigma|_{\Sigma\setminus W} \) is homotopically equivalent to \( \Sigma \setminus \Sigma|_{\Sigma\setminus W}, \) the exact sequence
\[ 0 \rightarrow \tilde{H}_d(\Sigma) \rightarrow \tilde{H}_d(\Sigma, \Sigma\setminus W|_{\Sigma\setminus W}) \rightarrow \tilde{H}_{d-1}(\Sigma|_{\Sigma\setminus W}) \rightarrow \cdots, \]
implies
\[ \text{rk} \; \tilde{H}_{d-1}(\Sigma|_{\Sigma\setminus W}) + 1 = \text{rk} \; \tilde{H}_{d-1}(\Sigma|_{\Sigma\setminus W}) + \text{rk} \; \tilde{H}_d(\Sigma) \geq \text{rk} \; \tilde{H}_d(\Sigma, \Sigma\setminus W|_{\Sigma\setminus W}) \geq \text{rk} H^0(\Sigma|_{\Sigma\setminus W}). \]

5. Applications to connectivity of manifolds

Let \( \Delta \) be a \((d-1)\)-dimensional simplicial complex on the vertex set \( V(\Delta) \). Recall the notion of banner complexes of [BV14]:
- A subset \( W \) of \( V(\Delta) \) is called **complete** if every two vertices of \( W \) form an edge of \( \Delta \).
- A complete set \( W \subseteq V(\Delta) \) is **critical** if \( W \setminus \{v\} \) is a face of \( \Delta \) for some \( v \in W \).
- We say that \( \Delta \) is **banner** if every critical complete set \( W \) of size at least \( d \) is a face of \( \Delta \).
- We define the **banner number** of \( \Delta \) to be
\[ b(\Delta) = \min \left\{ b : \text{lk}_\sigma \Delta \text{ is banner or the boundary of the 2-simplex} \right\}, \]
where the **degree** of a face is the maximal cardinality of a facet containing it.

Note that our notions of banner complexes and banner numbers are slightly more general then the ones introduced in [BV14]. However, if the complex is pure the definitions coincide.

**Lemma 9.** Let \( \Delta \) be a \((d-1)\)-dimensional simplicial complex.
(a) If \( \sigma \) is a face of degree \( d \) in \( \Delta \), then \( b(\text{lk}_\sigma \Delta) \leq \max\{0, b(\Delta) - \#\sigma\} \).
(b) If \( \Delta \) has nontrivial top-homology and \( b(\Delta) < d - 2 \), then every induced subcomplex of \( \Delta \) having nontrivial \((d-2)\)-homology has at least \( 2d - 2 - b(\Delta) \) vertices.

**Proof.** The part (a) is clear from the definition. For claim (b), let us first show that, if \( \Delta \) is banner, then every induced subcomplex \( \Gamma \) of \( \Delta \) such that \( \tilde{H}_{d-2}(\Gamma) \neq 0 \) has at least \( 2d - 2 \) vertices by induction on \( d \). If \( d = 3 \), this is clear because \( \Delta \) is flag.

Let \( d > 3 \). We may assume that no induced subcomplex of \( \Delta \) has a nontrivial \((d-1)\)-dimensional cycle: indeed, such a subcomplex is forced to have dimension \( d - 1 \), so it would be banner and we could replace \( \Delta \) with it. Furthermore, we may assume that \( \Gamma \) is a minimal induced subcomplex with the property
Theorem 12. Let $\Delta$ be a pure $(d - 1)$-dimensional complex with nontrivial top-homology. If $b(\Delta) < d - 2$, then $\mathbb{k}[\Delta]$ satisfies the property $\mathfrak{B}_{d-b(\Delta)-1}$.

Proof. Notice that the regularity of $\mathbb{k}[\Delta]$ is $d$, since $\Delta$ has a nontrivial top-homology. If $b_{i+d-1}(\mathbb{k}[\Delta]) \neq 0$, by Hochster’s formula there exists a subset $W \subseteq V(\Delta)$ of cardinality $i + d - 1$ such that $\Delta_W$ supports a nontrivial $(d - 2)$-cycle. By part (b) of Lemma 9, thus:

$$i \geq d - b(\Delta) - 1.$$ □

Theorem 12. Let $\Delta$ be a pure $(d - 1)$-dimensional minimal complex. Then the underlying graph of $\Delta$ is $(2d - b(\Delta) - 2)$-connected.

Proof. If $b(\Delta) = d - 2$, then it is easy to see that $\mathbb{k}[\Delta]$ satisfies $\mathfrak{B}_1$. By Lemma 8 and Hochster’s formula, $\mathbb{k}[\Delta]$ satisfies also the property $\mathfrak{A}$. Therefore, the result follows from Theorem 6.

If $b(\Delta) < d - 2$, by Lemmata 8 (together with Hochster’s formula) and 11, $\mathbb{k}[\Delta]$ satisfies the properties $\mathfrak{A}$ and $\mathfrak{B}_{d-b(\Delta)-1}$. Therefore, the result follows from Theorem 6. □

Corollary 13. Let $\Delta$ be a flag (or more generally banner) $(d - 1)$-dimensional minimal cycle. Then the underlying graph of $\Delta$ is $(2d - 2)$-connected.

Proof. If $\Delta$ is a banner complex, then $b(\Delta) = 0$. □

References


